Treball de Fi de Grau per a la obtenció del Grau en Matemàtiques i Grau en Física

Closing the loop

Periodic orbits of some non-autonomous dynamical systems with applications to fluid dynamics

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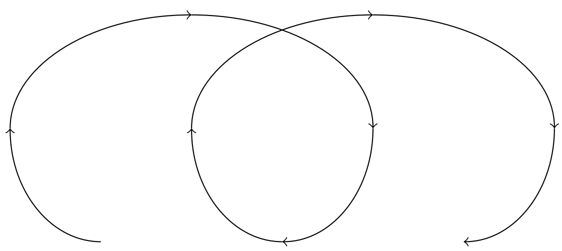

Bellaterra, 26th June 2012

Abstract

The equations of motion of the particle trajectories within the framework of linear irrotational water wave theory are a fully non-linear non-autonomous equations system for which no explicit solutions are available. Recent developments done by A. CONSTANTIN *et. al.* show, using a non-autonomous variable change and some phase-plane analysis, that the particle paths are not closed, but an actual net forward displacement, termed *Stokes drift*, is to be found. In the present work these results are firstly presented and discussed, and secondly extended into a more general model. After that, the same technique is used to find similar non-autonomous differential equations systems which do have periodic orbits.

Contents

Chapter 1

Introduction

> *"...the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place."*
>
> Leonardo da Vinci

Waves count among the most studied phenomena both in mathematics and in physics. It is a widely known fact that when we throw a stone in a lake or watch the surface waves on the sea, contrarily to what appears to us, it is not water what we see traveling, but only a shape pattern, a disturbance that propagates through it. In other words, the rapid motion of a wave is the product of a much slower motion of the substance through which it travels. We will restrict our attention to gravitational water waves, which are waves formed in water under a constant gravitational force.

The propagation of such waves can be described to a great accuracy without major problems. However, the equations of motion of each fluid particle constitute a fully non-linear non-autonomous ordinary differential equations system for which explicit solutions are not available. It is known that the fluid particles move slightly upwards and downwards, forwards and backwards, as the wave passes through them, but other aspects like whether their trajectories, or *particle paths*, are closed or not, are more difficult to answer. A classical approach to this problem, to be found in most books on fluid mechanics (cf. [1], [10], [12] and [16]), is to linearize the governing equations, obtaining elliptic and circular closed paths -depending on the boundary conditions-, degenerating at the sea bottom. However, G. G. Stokes already observed in 1847 that [17]:

> *"It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of propagation of the waves."*

Recent developments on the subject done by A. Constantin *et. al.* (cf. [3], [4], [7] and [8]) show that Stokes was right indeed, so there is an actual net forward displacement, termed *Stokes drift*. This modern approach is based on a smart variable change, which makes the equations system autonomous, thus letting phase-plane analysis techniques be used. The system can be shown to have no periodic orbits unless the surface is flat, so there are no closed particle paths.

The aim of this work is double: on the one hand, to present and examine such recent developments for irrotational water waves, both in the case of finite-depth water and of deep-water waves. On the other hand, to extend the main results to a more general system, as well as to see whether the same technique can be used to find similar non-autonomous equations systems for which there are indeed periodic orbits.

The work is structured as follows. In Chapter 2 the main mathematical model is presented. We shall permit us to start with a review of some basic notions of fluid dynamics and continue by setting out appropriate boundary conditions. We end by doing a non-dimensionalization and rescaling of variables, restricting us to the linear water waves regime. In Chapter 3 the developments of CONSTANTIN *et. al.* are presented and examined. We derive a fluid velocity field, we discuss the classical approach and we introduce the variable change. We also reproduce the phase-plane analysis and enunciate and prove the main results, both for finite-depth and deep-water. Chapter 4 contains all the original work and is divided in two sections. In the first we bring together the two situations as particular cases of a more general system depending on an arbitrary function fulfilling certain conditions and we extend the previous results to this system. In particular we show that there is a forward drift and that it is strictly decreasing with depth. In the second we look for similar non-autonomous systems, which can be shown to have periodic orbits using the same technique. We give three examples: one with a constant in depth but still positive forward drift, one for which existence of periodic orbits can be proven, and one for which all orbits are periodic.

Chapter 2

Notions of Hydrodynamics

"Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."

J. Wolfgang von Goethe

In this chapter the mathematical model for our study of gravitational water waves is going to be set. As a starting point, we will deduce the basic equations of inviscid fluid mechanics -namely the equation of *mass conservation* and *Euler's equations of motion*- from general principles, as well as adapt them to the analysis of water waves through some assumptions and simplifications -which will select the water-wave problem from all other possible solutions of the equations-. Following this, convenient boundary conditions will be stated so that our calculations become realistic and a non-dimensionalisation and scaling of variables is going to be made for easier mathematical manipulation.

2.1 The Governing Equations of Fluid Mechanics

We may start with some general considerations about fluid dynamics. If the reader is already familiar with it, we suggest to skip to §2.2. A fluid is normally defined as a *material which flows* (cf.[16]), but that is somehow too general to start a rigorous study with. In this work we are going to deal with liquids -actually almost exclusively with *water*-, which of course is matter and therefore essentially constituted by molecules, atoms, etc. As a consequence, one could think that any possible description of the motion and interaction with external forces of the fluid must involve quantum-mechanical methods, as well as take into account that most of the mass is condensed in the atomic nuclei and therefore almost all the space occupied by the fluid is empty.

However, this considerations become only important at a microscopic level -dimensions around $1\text{Å} = 10^{-10}\ m$- or smaller, and since we will be dealing with much bigger dimensions we may ignore this *discrete* character of matter -which is studied in quantum and statistical mechanics- and regard the fluid as being a continuum. That is the so called *continuum hypothesis.* We are therefore going to study the *average* manifestation of molecular forces, and describe only the macroscopic or gross behaviour of the fluid.

As a consequence of this, we are going to take a fixed Cartesian reference frame, and describe the state of the fluid through continuous functions of position $\mathbf{x} \equiv (x, y, z)$ and time t -y is taken to point upwards-, which are the only independent variables. For example the *pressure* function $P(\mathbf{x}, t)$, which describes the average force per unit area exerted to the boundaries of a volume of liquid due to the constant collisions of molecules. Another important example is the density function $\rho(\mathbf{x}, t)$, which describes the mass distribution per unit volume.

2.1.1 Fluid description

The main goal of fluid dynamics is to be able to fully describe the state of the fluid, that is, to know all its properties -like the pressure or the density- at every point $\mathbf{x}$ and at every time t. In this work we will not only assume that these functions are continuous but also of class $\mathcal{C}^1$.

One of the most important properties of a fluid state is the fluid velocity $\mathbf{u} \equiv (u, v, w)$, which of course is a function $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ -and therefore a time-dependent vector field-. This field is usually represented by *streamlines*, which are lines tangent to the fluid velocity at every point. In particular, if $\mathbf{u}(\mathbf{x}, t)$ changes with time, so do the streamlines. Another important definition is:

Definition 2.1 (***Particle path***)**.** *Let us consider a fluid particle -an infinitesimally small volume of liquid- which at time t_0 is at some point $\mathbf{x}_0$. Then its* particle path $\mathbf{x}(t)$ *is defined as the trajectory described by the fluid particle as time goes on, that is, the solution of the initial value problem*

$$\begin{cases} \dfrac{d}{dt}\mathbf{x}(t) &= \mathbf{u}(\mathbf{x}(t), t) \\ \mathbf{x}(t_0) &= \mathbf{x}_0. \end{cases} \tag{2.1}$$

We observe that if the fluid velocity field does not change with time, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})$ -which is called *steady flow*-, then the particle paths and the streamlines coincide. The concept of *particle path*, as well as their properties -whether they are closed or not, their shape, their periodicity...- is going to be the main topic of this work, as we said in the introduction.

Given a fluid property $f(\mathbf{x}, t)$, we may be interested in its time evolution. The first temptation would be to simply calculate $\frac{\partial f}{\partial t}$, but this will only give us the evolution of this property in a fixed space point. Since the water occupying this fixed space point changes continuously, in most cases the information we are going to get will not be the one we are looking for -let us consider for example that we are analysing the effect of some force acting on the fluid-. The concept of particle path lets us solve this problem through the following mathematical tool:

Definition 2.2 (***Material derivative***)**.** *Given a fluid property $f = f(\mathbf{x}, t)$, we define its* total *or* material derivative *as*

$$\frac{Df}{Dt} := \frac{df(\mathbf{x}(t), t)}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial t} = \nabla f \cdot \mathbf{u} + \frac{\partial f}{\partial t},$$

which gives us the evolution of f on a fixed particle fluid -since we take the particle path as the spacial variable-. Sometimes this derivative is regarded as the "derivative following the fluid".

For example, fluids like water are regarded to be *inviscid* (cf. [10]), which means that the density of the fluid does not change with time, that is,

$$\frac{D\rho}{Dt} = 0. \tag{2.2}$$

2.1.2 Mass conservation

Now we are going to derive the equation of *mass conservation* for a fluid, which can be found in any text on fluid mechanics (cf. [1], [10], [12], [16]). Let us imagine an arbitrary volume V within the fluid, fixed with respect to our reference frame and bounded by a closed surface ∂V. One the one hand, since V is fixed, the fluid in motion may cross the surface ∂V, both inwards and outwards, so that if the rate of change of mass in V is given by

$$\frac{d}{dt}\int_V \rho \, dV,$$

where $\rho = \rho(\mathbf{x}, t)$ represents the density and $dV = dx\,dy\,dz$ is the element of volume. On the other hand, the amount of water entering V is given by

$$-\oint_{\partial V} \rho\,\mathbf{u}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is the normal vector pointing outwards of ∂V and dS is the element of surface. Assuming that no liquid is created or destroyed in any point, we must have

$$\frac{d}{dt}\int_V \rho \, dV = -\oint_{\partial V} \rho\,\mathbf{u}\cdot\mathbf{n}\,dS. \tag{2.3}$$

Using Gauss' *divergence theorem*, together with the fact that $\rho \in \mathcal{C}^1$ and V is fixed -no time dependence-, we may rewrite (2.3) as

$$\frac{d}{dt}\int_V \rho \, dV + \int_V \nabla\cdot(\rho\,\mathbf{u})\,dV = 0 \quad \longrightarrow \quad \int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u})\right) dV = 0. \tag{2.4}$$

Since V was arbitrary, for (2.4) to be true for every V, we must have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u}) = 0$$

which is called *continuity equation*. Using now the vector identity $\nabla\cdot(\rho\mathbf{u}) = \rho(\nabla\cdot\mathbf{u}) + \nabla\rho\cdot\mathbf{u}$, the following holds

$$\frac{D\rho}{Dt} + \rho\,(\nabla\cdot\mathbf{u}) = 0.$$

Finally, if the fluid in inviscid -equation (2.2)-, we get

$$\nabla\cdot\mathbf{u} = 0, \tag{2.5}$$

which is known as the *equation of mass conservation* for inviscid fluids, and is totally independent of whether the flow is steady or not.

2.1.3 Euler's equations of motion

Until now, we have only stated some properties of the fluids' velocity field, but no dynamics have been yet described, that means, no considerations on the causes of the fluids' movement have been made. Fluid dynamics can be regarded as classical physics, and thus its dynamics are deterministically described by *Newton's second law*,

$$\mathbf{F} = m\,\mathbf{a}, \tag{2.6}$$

where $\mathbf{F}$ is the net external force, m is the mass and $\mathbf{a}$ is the acceleration. But how does actually equation (2.6) apply to a fluid?

Consider now, as we did before, an arbitrary volume V within the fluid, and its boundary closed surface ∂V. On the one hand, the force exerted by the surrounding fluid across any surface element δS is, by the definition of pressure, $P\mathbf{n}\,\delta S$, where $\mathbf{n}$ is again the outward normal vector. Thus, the net force exerted on V is

$$-\oint_{\partial V} P\,\mathbf{n}\,dS = -\int_V \nabla P\,dV,$$

where the *gradient theorem* was used. Since we are going to deal exclusively with gravity waves -waves formed by the presence of a constant gravitational force $\mathbf{P} = m\mathbf{g}$, being $\mathbf{g} = (0, -g, 0)$ the gravitational acceleration constant-, no other forces are to be considered.

On the other hand, the acceleration on a fluid is understood as the acceleration of every fluid element, that is, $\frac{D\mathbf{u}}{Dt}$. Therefore, equation (2.6) applied to V is

$$\int_V \rho\,\mathbf{g}\,dV - \int_V \nabla P\,dV = \int_V \rho\frac{D\mathbf{u}}{Dt}\,dV \quad\longrightarrow\quad \int_V \left(\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla P - \mathbf{g}\right) dV = 0,$$

and as we did before, this implies that

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla P - \mathbf{g} = 0,$$

which is known as the *Euler's equations of motion.*

Combining the two main equations of this section in an explicit form we have

$$\left\{\begin{array}{rcl} u_t + uu_x + vu_y + wu_z &=& -\frac{1}{\rho}P_x \\ v_t + uv_x + vv_y + wv_z &=& -\frac{1}{\rho}P_y - g \\ w_t + uw_x + vw_y + ww_z &=& -\frac{1}{\rho}P_z \\ u_x + v_y + w_z &=& 0, \end{array}\right. \tag{2.7}$$

where the subindices stand for partial derivatives.

2.2 Boundary Conditions

We may now set out the boundary conditions for our problem. As we already said, we will be dealing with gravity water waves. We are going to study their behaviour far away from the coast. Moreover, we will only consider two-dimensional waves: that is, we let x point in the direction of propagation of the wave -perpendicular to the crest line- and y upwards, and we consider that there is no difference in the behaviour along the z axis so this dimension can be eliminated from our analysis ($\mathbf{x} = (x, y)$), that means, only one cross-section of the flow will be taken into account, assuming that all of them look the same.

Our typical scenario is depicted in Fig. 2.1. The average height is denoted by b and the actual shape variation is given by a function $\eta(\mathbf{x}, t)$ so that the interface between water and air is going to be $y = \eta(x, t) + b$. We define the *fluid domain* as

$$\mathcal{D} := \{\ (x, y) \in \mathbb{R}^2 \mid a < y \leq \eta(x, t) + b\ \}.$$

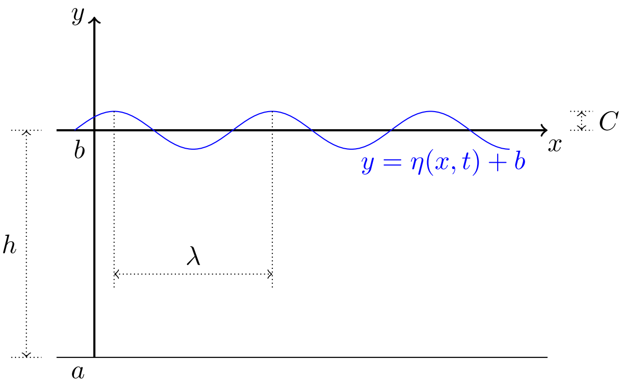

Figure 2.1: *Cross-section of the flow. The wave propagates in the positive direction of x.*

Dynamic boundary condition

Since the changes in pressure of the air lying above the fluid domain do not have a relevant effect on the fluid motion, we decouple the motion of the former and the latter through the *dynamic boundary condition*

$$P = P_{at} \qquad \text{on } y = \eta(x,t) + b, \tag{2.8}$$

where P_{at} stands for the atmospheric pressure and is taken to be constant.

Kinematic boundary conditions

The fluid domain boundary ∂D consists of two disjoint parts: the free surface and the bed. We want the particles lying in these two parts to stay there all the time. This means that the material derivative of each part's defining equations must cancel, which leads to the *kinematic boundary conditions*:

- *Free surface:* The defining equation is $S_1(\mathbf{x}, t) = 0$ where $S_1(\mathbf{x}, t) := y - \eta(x,t) - b$. Thus:

$$\frac{D}{Dt} S_1(\mathbf{x}, t) = 0 \quad \longrightarrow \quad v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(x,t) + b.$$

- *Bed:* The defining equation is $S_s(\mathbf{x}, t) = 0$ where $S_s(\mathbf{x}, t) := y - a$. Thus:

$$\frac{D}{Dt} S_2(\mathbf{x}, t) = 0 \quad \longrightarrow \quad v = 0 \quad \text{on} \quad y = a.$$

2.3 Non-Dimensionalisation and Scaling of the Variables

In this section we are going to adapt the governing equations (2.7) and the boundary conditions (2.8),(2.9) for a better mathematical handling, following mostly [3], [4], [5]. First of all we are going to non-dimensionalise them in order to get rid of most constants, and after that we will scale them in terms of two typical parameters which will lead us to a final linearisation of the equations.

2.3.1 Non-Dimensionalisation of the variables

Density

According to J. Lighthill (cf. [14]), the water density in seas, lakes and oceans is a function only of the pressure, the temperature and the salinity, but the first two account only for a 0.5%, 0.2% of variation and the third one up to 4% in big oceans, so in this work we may regard it as a constant. Actually, for the sake of simplicity we will further take $\rho = 1$.

Pressure

Since we are dealing with liquids, and in particular with gravity waves, we must take into account the effects of gravity on the fluid pressure. If the fluid were to be at rest, the pressure distribution due to gravity $\mathbf{g}$ would be given by the well-known *hydrostatic pressure distribution* expression ([12] and many others)

$$\frac{dP_h}{dy} = -\rho\, g = -g,$$

which together with the boundary condition $P = P_{at}$ at $y = b$ leads to: $P_h = P_{at} + g(b - y)$. Thus, if we want to take into account only the effects of fluid motion relative to the hydrostatic scenario, we may define a *non-dimensional pressure* p so that

$$P(\mathbf{x}, t) = P_h(y) + p\, b\, y = P_{at} + g\,(b - y) + g\, h\, p,$$

where $h := b - a$. The factor $g\, h$ of p is actually not necessary, but will be helpful when we apply further non-dimensionalisations.

Longitudinal variables

h is the typical or average depth. Similarly, we let λ be the typical wavelength and C the typical wave amplitude. We redefine

$$x \;\mapsto\; \lambda x \qquad y \;\mapsto hy \qquad \eta \;\mapsto\; C\eta \tag{2.10}$$

where this is to be understood as "where we used to write x we shall now write λx", so that (x, y) will now be non-dimensionalised.

Speed and time variables

The fact that for water of a given depth h the greatest possible wave speed is $\sqrt{gh}$ (cf. [10], [14]) suggests also a natural scale for non-dimensionalising the speed variable u (corresponding to direction x) and consequently a time scale $\lambda/\sqrt{gh}$. Special care must be taken with the vertical velocity v so that the mass conservation equation (2.5) remains valid, since now the spacial derivatives will also be scaled and the scale for each direction is different. At the end of the day we will have

$$u \;\mapsto\; \sqrt{gh}\, u \qquad v \;\mapsto\; \frac{h\sqrt{gh}}{\lambda}\, v \qquad t \;\mapsto\; \frac{\lambda}{\sqrt{gh}}\, t. \tag{2.11}$$

2.3.2 Scaling the governing equations

After the non-dimensionalisation our system of equations (2.7), together with the boundary conditions (2.8),(2.9), becomes:

$$\begin{cases} u_t + uu_x + vu_y &= -p_x \\ \delta^2(v_t + uv_x + vv_y) &= -p_y \\ u_x + v_y &= 0 \\ p &= \varepsilon\eta & \text{on} \quad y = \tilde{b} + \varepsilon\eta \\ v &= \varepsilon(\eta_t + u\eta_x) & \text{on} \quad y = \tilde{b} + \varepsilon\eta \\ v &= 0 & \text{on} \quad y = \tilde{a}, \end{cases} \tag{2.12}$$

where the following constants are defined,

$$\delta := \frac{h}{\lambda} \qquad \varepsilon := \frac{C}{h} \qquad \tilde{a} = \frac{a}{h} \qquad \tilde{b} = \frac{b}{h}.$$

The constants $\tilde{a}, \tilde{b}$ correspond simply to the depth and mean flat surface height after the non-dimensionalisation of y. The constant ε is called the *amplitude parameter* and δ is the *shallowness parameter*. These two parameters have a concrete physical meaning:

1. *The amplitude parameter ε:* It gives us information on how "big" the waves are, that is, how relevant the changes in time due to the surface wave $\eta(\mathbf{x}, t)$ are in comparison to the total depth. In this work we will think of the limit $\varepsilon \to 0$, which is called *linear water waves*, and corresponds to a "small disturbance of the underlying flow" (cf. [7]).

2. *The shallowness parameter δ:* It tells us the relative importance of the wavelength in comparison to the typical height. We may distinguish between *long* or *shallow* waves, with $\delta \to 0$, and *deep-water* waves, with $\delta \to \infty$.

In particular, they help us select the important terms of equations (2.12) and neglect the small ones. We will first do the following *scaling*:

$$p \mapsto \varepsilon p \qquad u \mapsto \varepsilon u \qquad v \mapsto \varepsilon v$$

so that equations (2.12) shall become:

$$\begin{cases} u_t + \varepsilon(uu_x + vu_y) &= -p_x \\ \delta^2[v_t + \varepsilon(uv_x + vv_y)] &= -p_y \\ u_x + v_y &= 0 \\ p &= \eta & \text{on} \quad y = \tilde{b} + \varepsilon\eta \\ v &= \eta_t + \varepsilon u\eta_x & \text{on} \quad y = \tilde{b} + \varepsilon\eta \\ v &= 0 & \text{on} \quad y = \tilde{a}. \end{cases} \tag{2.13}$$

Finally, we linearise the equations by letting $\varepsilon \to 0$ and therewith obtain:

$$\begin{cases} u_t &= -p_x \\ \delta^2 v_t &= -p_y \\ u_x + v_y &= 0 \\ p &= \eta & \text{on} \quad y = \tilde{b} \\ v &= \eta_t & \text{on} \quad y = \tilde{b} \\ v &= 0 & \text{on} \quad y = \tilde{a}. \end{cases} \tag{2.14}$$

Chapter 3

Particle paths in linear water waves

"What we know is a drop, what we do not know is an ocean."

Sir Isaac Newton

In this chapter we will present and examine the recent developments done by A. Constantin, M. Ehrnström and others. We will first look for travelling solutions of the governing equations derived in the previous chapter (2.14) in two different cases:

- Finite-depth water waves
- Deep-water waves

assuming in both cases irrotationality -absence of vorticity-. Under *travelling solution* we understand solutions as waves travelling at some speed $c > 0$, that is, with an (x, t)-dependence in the form $x - ct$. Consequently, after the non-dimensionalisation and scaling of variables done in §2.3, it may be logical to take as an *Ansatz* that the surface waves are of period one, that is $\eta(x, t) = \cos[2\pi(x - ct)]$, which corresponds to the fundamental Fourier mode. We will compare the classical results (to be found in any book on fluid mechanics, like [1], [10], [12], [16]) with the results of A. Constantin *et. al.* (cf. [3], [4], [7], [8]).

3.1 Finite-depth water waves

This section deals with the simplest physical situation (see Fig. 2.1), namely with $h < \infty$ and *irrotational* water (see below). For simplicity we will take $a = \tilde{a} = 0$ and $b = h$ so $\tilde{b} = 1$. Let us first introduce the concept of vorticity.

3.1.1 The concept of vorticity

We define the vorticity as follows:

Definition 3.1 (***Vorticity***)**.** *Given a fluid flow and let* $\mathbf{u}$ *be its velocity field, we define the vorticity as*

$$\boldsymbol{\omega} := \nabla \wedge \mathbf{u}.$$

In particular, if the flow is two-dimensional, $\mathbf{u}(\mathbf{x}, t) = (u(x, y, t), v(x, y, t), 0)$ *and* $\boldsymbol{\omega} = (0, 0, \omega)$ *where*

$$\omega = v_x - u_y.$$

A flow is called irrotational if $\boldsymbol{\omega} = 0$.

From a physical point of view, the vorticity represents *"twice the average angular velocity of two short fluid line elements that happen, at that instant, to be mutually perpendicular"* (cf.[1]). It is however important to make clear that it only refers to the *local* spin or whirl of the fluid. A typical example of this is the following: consider a two-dimensional flow confined in a ring $r_1 < r < r_2$, where (r, φ) refer to the polar coordinates, with velocity field $\mathbf{u} = Cr^{-1}\mathbf{e}_\varphi$, C a constant. Then although the global fluid motion consists of a rotation around the centre of the ring, the vorticity is

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = (0, 0, \omega) \qquad \text{where} \quad \omega = \frac{1}{r}\frac{\partial}{\partial r}(ru_\varphi) - \frac{1}{r}\frac{\partial u_r}{\partial \varphi} = 0,$$

so it is an irrotational flow.

Concerning the physical relevance of the assumption of irrotationality in water waves, according to [14] if the waves enter a region of still water, the assumption is realistic. Moreover, the following result applies:

Theorem 3.2 (**Cauchy-Lagrange**). *Let an inviscid, incompressible fluid of constant density move in the presence of a conservative conservative body force (for example gravity). Then if a portion of the fluid is initially in irrotational motion, that portion will always be in irrotational motion.*

This theorem, which is a consequence of Kelvin's *circulation theorem* (cf. [1]), implies that all initially irrotational flow will remain irrotational at all times, and therefore it makes sense to study this particular case.

3.1.2 Fluid velocity field

Proposition 3.3. *A solution of the system (2.14) for irrotational finite-depth water is*

$$\begin{aligned} u(x,y,t) &= \frac{2\pi c\delta}{\sinh(2\pi\delta)}\cosh(2\pi\delta y)\cos[2\pi(x-ct)] \\ v(x,y,t) &= \frac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta y)\sin[2\pi(x-ct)] \\ p(x,y,t) &= \frac{1}{\cosh(2\pi\delta)}\cosh(2\pi\delta y)\cos[2\pi(x-ct)] \end{aligned} \tag{3.1}$$

where

$$c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}.$$

Proof. Throughout this work we will assume that every function is regular enough for Schwartz's theorem to apply. We start by looking at the boundary condition $v = \eta_t$ on $y = 1$, which suggests a function v of the form $v(x, y, t) = F(y)\sin[2\pi(x - ct)]$. Deriving now the equation $u_t = -p_x$ with respect to x and y we get $u_{txy} = -p_{xxy}$. On the one hand, since $u_x + v_y = 0$, we have $u_{txy} = -v_{tyy}$. On the other hand, deriving the equation $\delta^2 = -p_y$ with respect to x twice we get $\delta^2 v_{txx} = -p_{yxx} = -p_{xxy}$. Thus, we obtain $\delta^2 v_{txx} = -v_{tyy}$. Applying this to our $v(x, y, t)$ leads to

$$\delta^2 c(2\pi)^3 F(y)\cos[2\pi(x-ct)] = 2\pi c F''(y)\cos[2\pi(x-ct)]$$

so

$$F''(y) - (2\pi\delta)^2 F(y) = 0, \tag{3.2}$$

which has the well-known general solution $F(y) = C_1 e^{2\pi\delta y} + C_2 e^{-2\pi\delta y}$.

The boundary condition $v = 0$ on $y = 0$ implies $C_2 = -C_1$, so we may write $v(x, y, t) = 2C_1 \sinh(2\pi\delta y)\times \sin[2\pi(x - ct)]$, and from $v = \eta_t$ on $y = 1$ we must have

$$2C_1 \sinh(2\pi\delta) \sin[2\pi(x - ct)] = 2\pi c \sin[2\pi(x - ct)] \quad \longrightarrow \quad C_1 = \frac{\pi c}{\sinh(2\pi\delta)}$$

so

$$v(x, y, t) = \frac{2\pi c}{\sinh(2\pi\delta)} \sinh(2\pi\delta y) \sin[2\pi(x - ct)].$$

From $u_x + v_y = 0$ we can find u as follows:

$$\begin{aligned} u(x, y, t) &= -\int v_y \, dx = -\int \frac{(2\pi)^2 c\delta}{\sinh(2\pi\delta)} \cosh(2\pi\delta y) \sin[2\pi(x - ct)] \, dx \\ &= \frac{2\pi c\delta}{\sinh(2\pi\delta)} \cosh(2\pi\delta y) \cos[2\pi(x - ct)]. \end{aligned}$$

Similarly, from $u_t = -p_x$ we get:

$$\begin{aligned} p(x, y, t) &= -\int u_t \, dx = -\int \frac{(2\pi c)^2 \delta}{\sinh(2\pi\delta)} \cosh(2\pi\delta y) \sin[2\pi(x - ct)] \, dx \\ &= \frac{2\pi c^2 \delta}{\sinh(2\pi\delta)} \cosh(2\pi\delta y) \cos[2\pi(x - ct)]. \end{aligned}$$

Finally, the boundary condition $p = \eta$ on $y = 1$ implies

$$\frac{2\pi c^2 \delta}{\sinh(2\pi\delta)} \cosh(2\pi\delta) \cos[2\pi(x - ct)] = \cos[2\pi(x - ct)] \quad \longrightarrow \quad c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta}$$

and p can be rewritten as

$$p(x, y, t) = \frac{1}{\cosh(2\pi\delta)} \cosh(2\pi\delta y) \cos[2\pi(x - ct)].$$

$\square$

As a consequence,

Corollary 3.4. *If we reverse the changes introduced in §2.3, the functions in (3.1) can be written as*

$$\begin{cases} \eta(x, t) = \varepsilon h \cos(kx - \Omega t) \\ u(x, y, t) = \varepsilon \Omega h \dfrac{\cosh(ky)}{\sinh(kh)} \cos(kx - \Omega t) \\ v(x, y, t) = \varepsilon \Omega h \dfrac{\sinh(ky)}{\sinh(kh)} \sin(kx - \Omega t) \\ P(x, y, t) = P_{at} + g(h - y) + \varepsilon g h \dfrac{\cosh(ky)}{\cosh(kh)} \cos(kx - \Omega t) \end{cases}$$

in terms of the physical variables, where

$$k = \frac{2\pi}{\lambda} \qquad \Omega = \sqrt{gk \tanh(kh)}.$$

Corollary 3.5. *The speed c of the linear wave is thus given by*

$$c = \frac{\Omega}{k} = \sqrt{g\frac{\tanh(kh)}{k}} \tag{3.3}$$

and the period T by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk\tanh(kh)}}.$$

This corollary gives us another physical insight into the meaning of the *shallowness parameter* δ. For *long* or *shallow waves*, $\delta = h/\lambda \to 0$ so $\lambda \to \infty$ -that is why they are called *long*-, and therefore $T \to \infty$. In this case, the wave speed c tends to $\sqrt{gh}$. However, for *deep-water waves*, that is when $\delta \to \infty$, $\delta \to \infty$ too -that is why it is called *deep-water*-, the wave speed tends to $\sqrt{gk} = \sqrt{\frac{g\lambda}{2\pi}}$ as we will see in the next section.

Now that the fluid velocity field $\mathbf{u}(\mathbf{x}, t)$ is completely known, we may start analysing what the particle paths look like. We recall that the particle paths are going to be the solutions $(x(t), y(t))$ of the system (2.1), which in this case will be

$$\begin{cases} \dfrac{dx}{dt} = M\cosh(ky)\cos(kx - \Omega t) \\ \dfrac{dy}{dt} = M\sinh(ky)\sin(kx - \Omega t) \end{cases} \tag{3.4}$$

where

$$M = \frac{\varepsilon\Omega h}{\sinh(kh)}, \tag{3.5}$$

for a given initial condition (x_0, y_0). The smoothness of the right-hand side, together with the fact that it is a bounded function -since y is bounded- guarantees the existence of a unique global solution (cf.[18]).

3.1.3 Classical particle paths

The classical approach to solve (3.4) consists of using approximations in terms of M, which is considered to be small in the limit of linear water waves ($\varepsilon \to 0$):

$$\begin{cases} \dfrac{dx}{dt} = M\cosh(ky_0)\cos(kx_0 - \Omega t) + \mathcal{O}(M^2) \\ \dfrac{dy}{dt} = M\sinh(ky_0)\sin(kx_0 - \Omega t) + \mathcal{O}(M^2) \end{cases} \tag{3.6}$$

where $\mathcal{O}(M^2)$ denotes terms of order M^2 or higher. Neglecting these, and integrating with respect to time we find

$$\begin{cases} x(t) \simeq x_0 - \dfrac{M}{\Omega}\cosh(ky_0)\sin(kx_0 - \Omega t) \\ y(t) \simeq y_0 + \dfrac{M}{\Omega}\sinh(ky_0)\cos(kx_0 - \Omega t). \end{cases} \tag{3.7}$$

The reader may recognize this result as the expression of an ellipse of horizontal semi-axis $L_x = \frac{M}{\Omega}\cosh(ky_0)$, vertical semi-axis $L_y = \frac{M}{\Omega}\sinh(ky_0)$ and centre (x_0, y_0). This means that up to a first-order approximation in M the particle paths are ellipses whose dimensions decrease, and flatten with depth:

$L_y/L_x = \tanh(ky_0) \to 0$ as $y \to 0$. In particular, for $y = 0$ they are straight horizontal lines (see Fig. 3.1). Another important observation is that the distance between their foci is

$$d = 2\sqrt{L_x{}^2 - L_y{}^2} = 2\sqrt{\left(\frac{M}{\Omega}\cosh(ky_0)\right)^2 - \left(\frac{M}{\Omega}\sinh(ky_0)\right)^2} = \frac{2M}{\Omega}$$

which remains constant with height.

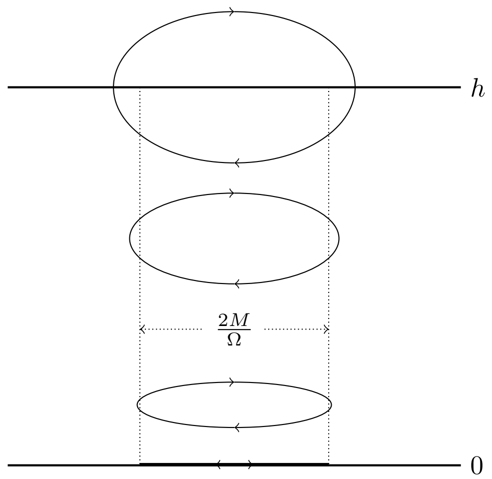

Figure 3.1: *Particle paths in the first-order approximation. The dimensions of the ellipses decrease with the depth while the distance between foci is maintained.*

3.1.4 A non-autonomous variable change

In the classical approximative approach that we have presented, particle paths are *closed*, or more precisely they are *periodic orbits* of the non-autonomous dynamical system (3.4). This has been the paradigm of linear water waves for many years, but in 2008 A. Constantin and G. Villari suggested in a paper[1] ([4]) a totally different approach based on the time-dependent variable change

$$X(t) := kx(t) - \Omega t \qquad Y(t) := ky(t) \tag{3.8}$$

which transforms system (3.4) into

$$\begin{cases} \dfrac{dX}{dt} & = \dfrac{\partial X}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial X}{\partial t} & = kM\cosh(Y)\cos(X) - \Omega & =: A(X,Y) \\ \dfrac{dY}{dt} & = \dfrac{dY}{dy}\dfrac{dy}{dt} & = kM\sinh(Y)\sin(X) & =: B(X,Y), \end{cases} \tag{3.9}$$

which is not only *autonomous*, but also *Hamiltonian*:

$$\begin{cases} \dfrac{dX}{dt} = H_Y \\ \dfrac{dY}{dt} = -H_X, \end{cases} \tag{3.10}$$

[1] The paper was actually written in 2005 during the attendance of both authors to the program "Wave Motion" at the Mittag-Leffler Institute in Stockholm, but was not actually published until 2008.

where

$$H(X,Y) = kM\sinh(Y)\cos(X) - \Omega Y, \tag{3.11}$$

so not only most of the tools from the qualitative theory of differential equations ([6], [15], [18]), but also from geometry ([2]) become available. In this work we will say that (3.4) and (3.9) are *conjugated* systems through the variable change (3.8).

The physical interpretation of the variable change (3.8) is, apart from a scaling, the transformation of a linear variable into a phase. This change is equivalent to a Galilean transformation of the reference frame with the constant velocity being that of the wave solution of the fluid velocity field. Since both $A(X,Y)$ and $B(X,Y)$ are 2π-periodic in the first variable, we can restrict our phase plane to the half-cylinder

$$\mathbb{D} := \{\, (X,Y) \in \mathbb{R}^2 \mid -\pi \le X \le \pi,\, 0 \le Y < \infty \,\} = C^1 \times [0,\infty),$$

where C^1 is the unit circle. Moreover, since $A(X,Y)$ is even in X and $B(X,Y)$ is odd, the system has mirror symmetry with respect to $X = 0$. We are now going to do a phase plane analysis of (3.9).

0- and ∞-isoclines

Definition 3.6. *The* 0-isocline (∞-isocline) *is the geometrical place of the points in* $\mathbb{D}$ *where the slope of the orbits is horizontal (vertical).*

Interpreting (3.9) as a vector field, the 0-isocline is characterized by the equation

$$B(X,Y) = k\,M\,\sinh(Y)\sin(X) = 0,$$

whose solutions are the lines $X = 0, \pm\pi$ and $Y = 0$ (see Fig. 3.2, colour green). This divides the cylinder into two equal regions -apart from the 0-isocline, which has zero measure-, one with $X \in (-\pi, 0)$, where the vector field goes downwards, and the other with $X \in (0,\pi)$ where the field goes upwards. In a similar way, the ∞-isocline is characterized by

$$A(X,Y) = kM\cosh(Y)\cos(X) - \Omega = 0. \tag{3.12}$$

Since all the constants are positive, this equation has no solutions for $|X| \ge \pi/2$ because then $\cos(X) \le 0$. For $X \in (-\pi/2, \pi/2)$, equation (3.12) implicitly defines a curve $(X,Y) = (X,\gamma(X))$ where

$$\gamma(X) := \operatorname{arccosh}\left(\frac{\Omega}{kM\cos(X)}\right).$$

The main features of this function are that it is even, $\gamma(X) \to +\infty$ when $X \to \pm\pi/2$ and its image is $[Y^*,\infty)$ with $Y^* = \operatorname{arccosh}(\Omega/kM) > 0$. This curve is represented in Fig. 3.2 with colour blue. It also divides $\mathbb{D}$ into two regions, this time unequal, such that above the curve the field goes to the right, and below to the left.

Phase-portrait

The confection of the phase-portrait of (3.9) is straightforward: the fact that the system is Hamiltonian implies that the function (3.11) is constant along the trajectories

$$\frac{dH}{dt} = \frac{\partial H}{\partial X}\frac{dX}{dt} + \frac{\partial H}{\partial Y}\frac{dY}{dt} = \frac{\partial H}{\partial X}\frac{\partial H}{\partial Y} - \frac{\partial H}{\partial Y}\frac{\partial H}{\partial X} = 0,$$

and therefore they are contained into the level sets of $H(X,Y)$. But since the system is two-dimensional, these level sets are curves, so they coincide with the trajectories. We recall some useful concepts and results of differentiable functions (cf. [2]):

Definition 3.7 (***Hessian matrix***). *Given an open set $\mathbb{D} \subseteq \mathbb{R}^n$ and a smooth function $H : \mathbb{D} \to \mathbb{R}$ of class $\mathcal{C}^2$, we define the* Hessian matrix *of H in the point $P \in \mathbb{D}$ as*

$$\mathcal{H}_H(P) = \left.\begin{pmatrix} H_{x_1,x_1} & H_{x_1,x_2} & \cdots & H_{x_1,x_n} \\ H_{x_2,x_1} & H_{x_2,x_2} & \cdots & H_{x_2,x_n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{x_n,x_1} & H_{x_n,x_2} & \cdots & H_{x_n,x_n} \end{pmatrix}\right|_P .$$

Definition 3.8 (***Singular point***). *Given an open set $\mathbb{D} \subseteq \mathbb{R}^n$ and a smooth function $H : \mathbb{D} \to \mathbb{R}$ of class $\mathcal{C}^2$, we say that a point $P \in \mathbb{D}$ is* singular *if $\nabla H(P) = 0$. Moreover, we say that P is degenerate if* $\det(\mathcal{H}_H(P)) = 0$ *and* non-degenerate *if not. If $\nabla H \neq 0$ we say that P is* regular.

Theorem 3.9 (***Morse's lemma***). *Given an open set $\mathbb{D} \subseteq \mathbb{R}^n$ and a smooth function $H : \mathbb{D} \to \mathbb{R}$ of class $\mathcal{C}^2$, let $P \in \mathbb{D}$ be a singular and non-degenerate point. Then there exists a coordinate system $(x_1, x_2, \ldots, x_n)$ in a neighbourhood $U \subset \mathbb{D}$ of P such that*

$$H(Q) = H(P) - {x_1}^2(Q) - ... - {x_k}^2(Q) + {x_{k+1}}^2(Q) + ... + {x_n}^2(Q), \quad \forall Q \in U,$$

where k is the number of eigenvalues of $\mathcal{H}_H(P)$ with negative real part. In other words, the function H will be locally equivalent to

$$x \mapsto F(P) - {x_1}^2 - ... - {x_k}^2 + {x_{k+1}}^2 + ... + {x_n}^2.$$

In particular, for $n = 2$ there are only three possibilities:

- $H(x) = H(P) - {x_1}^2 - {x_2}^2$
- $H(x) = H(P) + {x_1}^2 + {x_2}^2$
- $H(x) = H(P) - {x_1}^2 + {x_2}^2$.

In the first two cases the level curves of H in U will be diffeomorphically equivalent to circles, and P will be a maximum and a minimum of H respectively. In the third case, we say that P is a saddle point.

From definition 3.8 it is clear that the singular points are precisely the intersections of the two isoclines. In this case, there is only one singular point, $P = (0, Y^*)$. If we calculate the Hessian matrix of H in this point:

$$\mathcal{H}_H(P) = \left.\begin{pmatrix} H_{XX} & H_{XY} \\ H_{YX} & H_{YY} \end{pmatrix}\right|_P = \begin{pmatrix} -kM\sinh(Y^*) & 0 \\ 0 & kM\sinh(Y^*) \end{pmatrix},$$

so P is non-degenerate and trivially has the eigenvalues $\pm kM\sinh(Y^*)$. From theorem 3.9, since $k = 1$, P is a saddle point. The curves implicitly given by $H(X,Y) = H(P)$ are called *separatrices* and divide the phase plane $\mathbb{D}$ into four regions with trajectories of qualitatively different behaviour (see Fig.3.2, colour red).

Since we are only interested in the *physically realistic* trajectories -that is, the ones that do not diverge at any point-, we must impose that the singular point is higher than the water height. Recalling the multiple variable changes, this condition may be expressed as:

$$Y \leq Y^* \underset{(3.8)}{\longrightarrow} ky \leq k(h + C) = kh(1+\varepsilon) \leq Y^* \underset{(3.5)}{\longrightarrow} \tag{3.13}$$

$$kh \leq kh(1+\varepsilon) \leq \operatorname{arccosh}\left(\frac{1}{\varepsilon}\frac{\sinh(kh)}{kh}\right) \longrightarrow \cosh(kh) \leq \frac{1}{\varepsilon}\frac{\sinh(kh)}{kh} \tag{3.14}$$

so the actual requirement is

$$\varepsilon \leq \frac{\tanh(kh)}{kh},$$

which is coherent with the linear water waves limit ($\varepsilon \to 0$).

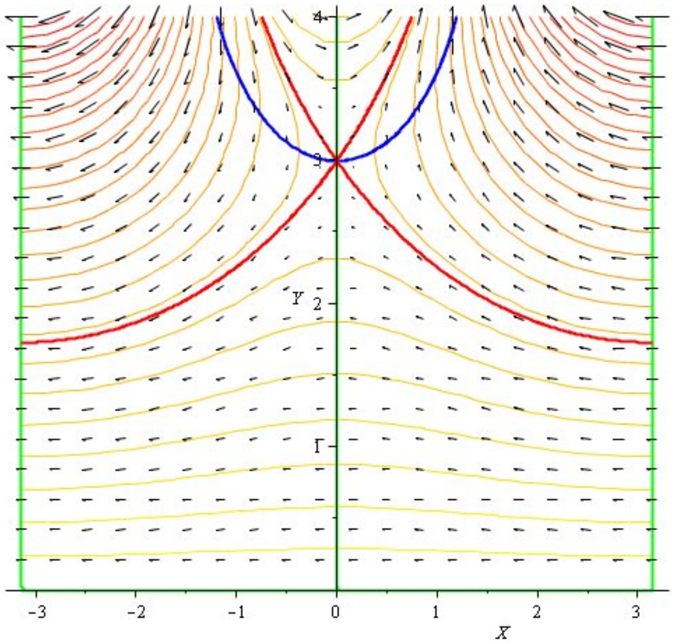

Figure 3.2: *Phase portrait of the system (3.9). The 0-isocline and $\infty-$isocline are represented in green and blue respectively. Their intersection is the singular point P_0. The red lines correspond to the separatrices, which divide the phase space into qualitatively different regions. We are only interested in the lower one, where the trajectories go from $X = \pi$ to $X = -\pi$.*

3.1.5 Main results

We are now in conditions to present the two main results of this approach:

Lemma 3.10. *Let (π, β) be the intersection point of the lowest separatrix with the line $X = \pi$, and $Y_\pi \in [0, \beta)$. Let also $(X(t), Y(t))$ be the solution of (3.9) with initial condition $(X(0), Y(0)) = (\pi, Y_\pi)$ and $\mathcal{T} = \mathcal{T}(Y_\pi)$ the time needed for this solution to intersect the line $X = -\pi$. Then the phase trajectory $(X(t), Y(t))$ corresponds -reversing the variable change (3.8)- to a periodic solution of (3.4) if and only if*

$$\mathcal{T}(Y_\pi) = \frac{2\pi}{\Omega}.$$

Proof. We stated before that the system (3.9) has mirror symmetry with respect to the line $Y = 0$. Thus, the phase trajectory $(X(t), Y(t))$ will intersect the line $X = -\pi$ at the point $(-\pi, Y_\pi)$, so $Y(\mathcal{T}) = Y_\pi$. This means that reversing (3.8), we have $y(\mathcal{T}) = y(0)$.

Let us start supposing that $\mathcal{T}(Y_\pi) = \dfrac{2\pi}{\Omega}$. Since the system (3.4) is non-autonomous, for a trajectory to be $\mathcal{T}$-periodic two conditions must be fulfilled: that after a time $\mathcal{T}$ the system is in the same point,

$(x(\mathcal{T}), y(\mathcal{T})) = (x(0), y(0))$ and that the r.h.s of (3.4) evaluated at time $t = \mathcal{T}$ is the same as at time $t = 0$. Indeed:

$$x(\mathcal{T}) - x(0) = \frac{X(\mathcal{T}) + \Omega\mathcal{T}}{k} - \frac{X(0)}{k} = \frac{-\pi + 2\pi}{k} - \frac{\pi}{k} = 0$$

and

$$[kx(\mathcal{T}) - \Omega\mathcal{T}] - [kx(0) - \Omega 0] = [X(\mathcal{T}) + \Omega\mathcal{T} - \Omega\mathcal{T}] - X(0) = -\pi - \pi = -2\pi,$$

which fulfils the condition because there are only 2π-periodic functions in $kx - \Omega t$ involved.

Conversely, let us now assume that there is a periodic solution of (3.4) of period $\tau > 0$. We know that $y(\tau) = y(0)$ and thus $Y(\tau) = Y(0)$ so $\tau = n\mathcal{T}$ for some $n \in \mathbb{N}$. We also know that $x(\tau) = x(n\mathcal{T}) = x(0)$ so $X(\tau) = X(0) - \Omega\tau = X(0) - n\Omega\mathcal{T}$. That implies

$$0 = X(\tau) - X(0) + n\Omega\mathcal{T} = X(n\mathcal{T}) - X(0) + n\Omega\mathcal{T} = (1-2n)\pi - \pi + n\Omega\mathcal{T} = -2n\pi + n\Omega\mathcal{T}$$

so

$$\mathcal{T} = \frac{2\pi}{\Omega}.$$

$\square$

The function $\mathcal{T}$ will be called *period function* throughout this work, although this name is sometimes used for other functions in the literature. The main result of [4] is:

Theorem 3.11. *There are no periodic orbits in (3.4).*

Proof. Using the previous lemma it suffices to prove that $\forall Y_\pi \in [0, \beta)$, $\mathcal{T}(Y_\pi) > \dfrac{2\pi}{\Omega}$. We will show this in two steps:

- $\mathcal{T}(0) > \dfrac{2\pi}{\Omega}$.
- For $Y_\pi \in (0, \beta]$, $\mathcal{T}(Y_\pi) > \mathcal{T}(0)$.

Starting with $Y(0) = 0$, we see that $dY/dt = B(X, Y) = B(X, 0) = 0$ so $Y = 0\ \forall t$. Thus, the only variations are in X. We have

$$\frac{dX}{dt} = kM\cos(X) - \Omega \quad \longrightarrow \quad \mathcal{T}(0) = \int_{\pi}^{-\pi} \frac{dX}{kM\cos(X) - \Omega} = \int_{-\pi}^{\pi} \frac{dX}{1 - \mu\cos(X)}$$

where

$$\mu = \frac{kM}{\Omega} = \frac{k\varepsilon\Omega h}{\Omega\sinh(kh)} = \varepsilon\frac{kh}{\sinh(kh)} \le \frac{\tanh(kh)}{kh}\frac{kh}{\sinh(kh)} = \frac{1}{\cosh(kh)} < 1.$$

Doing now the variable changes $u = \tan(X/2)$ and $v = \sqrt{\frac{1+\mu}{1-\mu}}u$ we have

$$\begin{aligned}\mathcal{T}(0) = \frac{1}{\Omega}\int_{-\infty}^{\infty} \frac{1}{1 - \mu\frac{1-u^2}{1+u^2}}\frac{2\,du}{1+u^2} = \frac{2}{\Omega\sqrt{1-\mu^2}}\int_{-\infty}^{\infty}\frac{dv}{1+v^2} = \\ \frac{2}{\Omega\sqrt{1-\mu^2}}\arctan(v)\Big|_{-\infty}^{\infty} = \frac{2\pi}{\Omega}\frac{1}{\sqrt{1-\mu^2}} > \frac{2\pi}{\Omega}.\end{aligned} \tag{3.15}$$

Let now $Y_\pi \in (0, \beta)$. If we call $Y_{\pi/2}$ the height of the intersection point of the trajectory with the line $X = \frac{\pi}{2}$, looking at Fig. 3.2 we see that for $X \in (-\frac{\pi}{2}, \frac{\pi}{2})$ the trajectory lies above the line $Y = Y_{\pi/2}$ while for $|X| > \frac{\pi}{2}$ it lies below. Thus

$$\frac{dX}{dt} = kM\cosh(Y)\cos(X) - \Omega \geq kM\cosh(Y_{\pi/2})\cos(X) - \Omega$$

and as a consequence

$$\mathcal{T}(Y_\pi) \geq \int_\pi^{-\pi} \frac{dX}{kM\cosh(Y_{\pi/2})\cos(X) - \Omega} = \frac{2\pi}{\Omega}\frac{1}{\sqrt{1-\mu'^2}} > \frac{2\pi}{\Omega},$$

using the same technique as above, but now with $\mu' = \dfrac{kM\cosh(Y_{\pi/2})}{\Omega} > \mu$. $\square$

3.1.6 Physical interpretation

We had seen in §3.1.3 that in a first approximation, the particle paths were ellipses -and thus closed-. However, we have proved that actually the trajectories of (3.4) are not periodic. A natural question to be asked now is: what do they look like then? Looking at the system (3.9) we see that the trajectories lying below the separatrix are periodic in the cylinder, because if we start a certain point (π, Y_π) -which we can assume, doing a time-shift if necessary-, we will end at the same height in the point $(-\pi, Y_\pi)$ after a time $\mathcal{T}$ -thus justifying the nomenclature *period function*-. Interpreting $X(t)$ as a phase and looking at the system 3.4 we see that during this time four different behaviours appear:

	$X \in (\frac{\pi}{2}, \pi)$	$X \in (0, \frac{\pi}{2})$	$X \in (-\frac{\pi}{2}, 0)$	$X \in (-\pi, -\frac{\pi}{2})$
$\frac{dx}{dt}$	< 0	> 0	> 0	< 0
$\frac{dy}{dt}$	> 0	> 0	< 0	< 0

Moreover, we have said that the initial and the final height are the same $y(0) = y(\mathcal{T})$, and in the table we see that actually this is the minimum height. Regarding the horizontal dimension, the theorem implies

$$x(\mathcal{T}) - x(0) = \frac{X(\mathcal{T}) - X(0)}{k} + \frac{\Omega\mathcal{T}}{k} = -\frac{2\pi}{k} + \frac{\Omega\mathcal{T}}{k} > 0$$

so the trajectories in the system (3.4) must look like in Fig. 3.3. In particular, there is a net forward *drift*, which is known as *Stokes drift* in the literature[2].

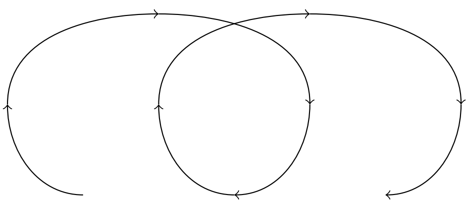

Figure 3.3: *Schematic representation of the actual particle paths in the system* (3.4).

[2]see [4] and references therein.

3.2 Deep-water waves

In this section we will deal with a similar case but we will think of $h \to \infty$ (so $\delta \to \infty$, the *deep-water waves* limit) and we will analyse what effects this change has on the particle paths.

3.2.1 Changes in the model

The set is now presented in Fig.3.4. We will now take $b = 0$ and think of $a \to -\infty$, $h \to \infty$. Thus, the fluid domain will now be

$$\mathcal{D} := \{ (x, y) \in \mathbb{R}^2 \mid y \leq \eta(x, t) \}.$$

This has important consequences in our model, since the whole non-dimensionalisation and scaling of the variables that we did in §2.3 was based on the assumption $h < \infty$. The general rule that we will apply will be to take $h = 1$ in this changes. In particular, y remains unchanged and

$$u \mapsto \sqrt{g}\, u \qquad v \mapsto \frac{\sqrt{g}}{\lambda} v \qquad t \mapsto \frac{\lambda}{\sqrt{g}} t \qquad \delta := \frac{1}{\lambda} \qquad \varepsilon := C. \tag{3.16}$$

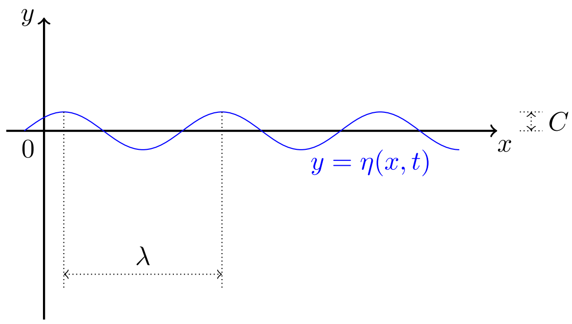

Figure 3.4: *Schematic scenario for deep-water waves.*

3.2.2 Fluid velocity field

Proposition 3.12. *The solution of the system (2.14) for irrotational deep water is*

$$\begin{aligned} u(x, y, t) &= 2\pi c\delta \exp(2\pi\delta y) \cos[2\pi(x - ct)] \\ v(x, y, t) &= 2\pi c \exp(2\pi\delta y) \sin[2\pi(x - ct)] \\ p(x, y, t) &= 2\pi c^2\delta \exp(2\pi\delta y) \cos[2\pi(x - ct)] \end{aligned} \tag{3.17}$$

where

$$c^2 = \frac{1}{2\pi\delta}.$$

Proof. The proof is essentially the same as in Proposition 3.1 but with different boundary conditions. Starting with the equation (3.2):

$$F''(y) - (2\pi\delta)^2 F(y) = 0,$$

and its general solution $F(y) = C_1 \exp(2\pi\delta y) + C_2 \exp(-2\pi\delta y)$, we must now impose that $F(y) \to 0$ when $y \to -\infty$, which implies $C_2 = 0$. From $v = \eta_t$ on $y = 0$ we get

$$C_1 \sin[2\pi(x - ct)] = 2\pi c \sin[2\pi(x - ct)] \quad \longrightarrow \quad C_1 = 2\pi c$$

so

$$v(x,y,t) = 2\pi c \exp(2\pi\delta y)\sin[2\pi(x-ct)].$$

Now $u_x + v_y = 0$ implies

$$\begin{aligned} u(x,y,t) &= -\int v_y\,dx = -\int (2\pi)^2 c\delta \sin[2\pi(x-ct)]\,dx \\ &= 2\pi c\delta \exp(2\pi\delta y)\cos[2\pi(x-ct)]. \end{aligned}$$

and $u_t = -p_x$ implies:

$$\begin{aligned} p(x,y,t) &= -\int u_t\,dx = -\int (2\pi c)^2 \delta \exp(2\pi\delta y)\sin[2\pi(x-ct)]\,dx \\ &= 2\pi c^2\delta \exp(2\pi\delta y)\cos[2\pi(x-ct)]. \end{aligned}$$

From the boundary condition $p = \eta$ on $y = 0$ we finally get

$$2\pi c^2\delta\cos[2\pi(x-ct)] = \cos[2\pi(x-ct)] \quad\longrightarrow\quad c^2 = \frac{1}{2\pi\delta}.$$

□

In a similar fashion as for finite-depth water:

Corollary 3.13. *If we reverse the changes introduced in §3.16, the functions in (3.17) can be written as*

$$\begin{cases} \eta(x,t) = \varepsilon\cos(kx-\Omega t) \\ u(x,y,t) = \varepsilon\Omega\exp(ky)\cos(kx-\Omega t) \\ v(x,y,t) = \varepsilon\Omega\exp(ky)\sin(kx-\Omega t) \\ P(x,y,t) = P_{at} - gy + \varepsilon g\exp(ky)\cos(kx-\Omega t) \end{cases}$$

in terms of the physical variables, where

$$k = 2\pi\delta = \frac{2\pi}{\lambda} \qquad \Omega = \sqrt{gk}.$$

Corollary 3.14. *The speed c of the linear wave is thus given by*

$$c = \frac{\Omega}{k} = \sqrt{\frac{g\lambda}{2\pi}}$$

and the period T by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk}}.$$

As expected, we recover the limit $\delta \to \infty$ of 3.3, which corresponds to *deep-water waves.* Now the dynamical system for the particle paths will be given by:

$$\begin{cases} \dfrac{dx}{dt} = M\exp(ky)\cos(kx-\Omega t) \\ \dfrac{dy}{dt} = M\exp(ky)\sin(kx-\Omega t) \end{cases} \tag{3.18}$$

where $M = \varepsilon\Omega$.

3.2.3 Classical particle paths

If we approximate the system (3.18) to first-order in M we get, in analogy to (3.6):

$$\begin{cases} \dfrac{dx}{dt} = M\exp(ky_0)\cos(kx_0 - \Omega t) + \mathcal{O}(M^2) \\ \dfrac{dy}{dt} = M\exp(ky_0)\sin(kx_0 - \Omega t) + \mathcal{O}(M^2) \end{cases} \tag{3.19}$$

and neglecting the terms of second order or higher and integrating with respect to time we get

$$\begin{cases} x(t) \simeq x_0 - \dfrac{M}{\Omega}\exp(ky_0)\sin(kx_0 - \Omega t) \\ y(t) \simeq y_0 + \dfrac{M}{\Omega}\exp(ky_0)\cos(kx_0 - \Omega t). \end{cases} \tag{3.20}$$

This equation correspond to a circle, so that for infinite depth the particle paths are *circular* to a first approximation, of radius $\frac{M}{\Omega}\exp(ky_0) = \varepsilon\exp(ky_0)$. This should come as no surprise, since for the finite-depth case the two semi-axis tended to be equal as $y_0 \to +\infty$ because $\tanh(ky_0) \to 1$ when $y_0 \to +\infty$. In particular we see that the water in the surface ($y = 0$) describe circles of radius ε, which is the amplitude of the wave η, and that these circles get smaller and smaller as we get deeper into the water, becoming punctual in the limit $y \to -\infty$.

3.2.4 Main results

A. Constantin *et al.* applied also the variable change (3.8) to the system (3.18) (cf. [3]), giving out the conjugated system

$$\begin{cases} \dfrac{dX}{dt} &= kM\exp(Y)\cos(X) - \Omega &=: A(X,Y) \\ \dfrac{dY}{dt} &= kM\exp(Y)\sin(X) &=: B(X,Y), \end{cases} \tag{3.21}$$

which is also Hamiltonian

$$H(X,Y) = kM\exp(Y)\cos(X) - \Omega Y.$$

Phase portrait

The phase portrait of (3.21) is very similar to the one of (3.9) (see Fig. 3.2). This time,

$$\mathbb{D} := \{\, (X,Y) \in \mathbb{R}^2 \mid -\pi \le X \le \pi,\, Y \le 0 \,\} = C^1 \times (-\infty, 0],$$

the 0-isocline will be only the lines $X = 0$ and $X = \pm\pi$ and the $\infty-$isocline will be a curve of the form $(X, \gamma(X))$ where $X \in (-\pi/2, \pi/2)$ -since all the constants are again positive- and

$$\gamma(X) := \log\left(\frac{\Omega}{kM\cos(X)}\right), \tag{3.22}$$

which also diverges to $+\infty$ when $X \to \pm\pi/2$. That implies that if there is a singular point, it must be in the line $X = 0$, so

$$0 = A(0, Y^*) = kM\exp(Y^*) - \Omega \quad \longrightarrow \quad Y^* = \log\left(\frac{\Omega}{kM}\right) = \log\left(\frac{1}{k\varepsilon}\right).$$

The Hessian matrix of H in this point $P = (0, Y^*)$ is

$$\mathcal{H}_H(P) = \begin{pmatrix} H_{XX} & H_{XY} \\ H_{YX} & H_{YY} \end{pmatrix}\Bigg|_P = kM\exp(Y)\begin{pmatrix} -\cos(X) & -\sin(X) \\ -\sin(X) & \cos(X) \end{pmatrix}\Bigg|_P = \begin{pmatrix} -\Omega & 0 \\ 0 & \Omega \end{pmatrix},$$

so it is again non-degenerate and a saddle point by virtue of theorem 3.9. There will be also four separatrices. As before, since we are only interested in the physically realistic trajectories, we must assure that P lies above the water surface, that is

$$Y \leq \log\left(\frac{1}{k\varepsilon}\right) \underset{(3.8)}{\longrightarrow} y \leq k\log\left(\frac{1}{k\varepsilon}\right) \longrightarrow \frac{1}{k\varepsilon} \leq \log\left(\frac{1}{k\varepsilon}\right).$$

It is enough to impose

$$\varepsilon < \frac{1}{ke}$$

since

$$k\varepsilon < \frac{1}{e} < 1 = \log(e) < \log\left(\frac{1}{k\varepsilon}\right).$$

And finally, the main result of [3]:

Theorem 3.15. *There are no periodic orbits in (3.18).*

Proof. The lemma (3.10) depends only on the variable change (3.8), so it is also valid for infinite depth water and thus it is enough to prove that, using the same notation as before,

$$\mathcal{T}(Y_\pi) > \frac{2\pi}{\Omega} \quad \forall Y_\pi \in (-\infty, \beta),$$

where (π, β) is the intersection point of the lowest right separatrix and the line $X = \pi$. To show this we will use a very similar reasoning as we did for the finite case. Given a $Y_\pi \in (-\infty, \beta)$, we call $Y_{\pi/2}$ the height of the intersection point of the trajectory with the line $X = \frac{\pi}{2}$, which is the same as that with the line $X = -\frac{\pi}{2}$. Since $\frac{dY}{dt} > 0$ for $X > 0$ and $\frac{dY}{dt} > 0$ for $Y < 0$, we know that the trajectory lies above $Y = Y_{\pi/2}$ for $X \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and below for $|X| > \frac{\pi}{2}$. As a consequence,

$$\frac{dX}{dt} = kM\exp(Y)\cos(X) - \Omega \geq kM\exp(Y_{\pi/2})\cos(X) - \Omega$$

so

$$\mathcal{T}(Y_\pi) \geq \int_\pi^{-\pi} \frac{dX}{kM\exp(Y_{\pi/2})\cos(X) - \Omega} = \int_{-\pi}^{\pi} \frac{dX}{\Omega - kM\exp(Y_{\pi/2})\cos(X)} =$$
$$\frac{1}{\Omega}\int_{-\pi}^{\pi} \frac{dX}{1 - \mu\cos(X)} = \frac{2\pi}{\Omega}\frac{1}{\sqrt{1-\mu^2}} > \frac{2\pi}{\Omega}$$

where the integral of (3.15) was used, provided that

$$\mu := \frac{kM\exp(Y_{\pi/2})}{\Omega} < \frac{kM}{\Omega} = k\varepsilon < \frac{1}{e} < 1.$$

□

The physical interpretation done in §3.1.6 naturally applies also here, so the orbits will also look like 3.3, although more "circle-like" than "ellipse-like".

Chapter 4

Looking for closed paths

"A singular disadvantage of the sea lies in the fact that after successfully surmounting one wave you discover that there is another behind it just as important and just as nervously anxious to do something effective in the way of swamping boats."

Stephen Crane

This is the main chapter of the work and contains all the original results. It deals both with the generalization of the systems we have already seen as well as with the possibility of having periodic orbits in similar systems. The two models of irrotational waves have been shown to have similar properties. We will see that they can be understood as particular cases of a more general model that has the same properties. After that we will look for similar non-autonomous systems, which can be shown to have periodic orbits. Three interesting examples will be given.

4.1 A general model for water waves

Looking at the dynamical systems describing the particle paths, both before (3.4), (3.18) and after (3.9), (3.21) the variable change, it is not difficult to see that they share a common form. Moreover, after the analysis of the phase portraits, we have concluded that both show a *forward drift*. At this point, the most natural question to be risen is whether their common properties are consequence of their common form, and therefore they can be thought as particular cases of a more general model. In this section we will give a positive answer to this question.

4.1.1 Phase-portrait analysis

Let $G : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$-function, fulfilling

- $\lim\limits_{Y \to a^+} G(Y) = 0$
- $G'(Y) > 0 \ \forall Y \in (a, b)$
- $G''(Y) > 0 \ \forall Y \in (a, b)$,

for some a, b which do need not to be finite.

We consider the following dynamical system

$$\begin{cases} \dfrac{dx}{dt} = M\, G'(ky)\, \cos(kx - \Omega t) \\ \dfrac{dy}{dt} = M\, G(ky)\, \sin(kx - \Omega t), \end{cases} \tag{4.1}$$

where M, k and Ω are real positive constants with $\Omega > k\,M$, as well as its conjugated system:

$$\begin{cases} \dfrac{dX}{dt} & = k\,M\, G'(Y)\, \cos(X) - \Omega & := A(X,Y) \\ \dfrac{dY}{dt} & = k\,M\, G(Y)\, \sin(X) & := B(X,Y) \end{cases} \tag{4.2}$$

which is also Hamiltonian:

$$H(X,Y) = k\,M\,G(Y)\,\cos(X) - \Omega\,Y.$$

Both systems have existence and uniqueness of solutions. Whatismore,

Remark 4.1. *If we make $G(Y) = \sinh(Y)$ for the finite-depth water case, or $G(Y) = \exp(Y)$ for the deep-water case, we recover (3.4), (3.18) from (4.1) and (3.9), (3.21) from (4.2). We also see that in both cases $G(Y)$ fulfils the required conditions.*

Following this remark, we define the domain of (4.2) to be

$$\mathbb{D} := \{\, (X,Y) \in \mathbb{R}^2 \mid -\pi \le X < \pi,\ a \le Y \le b \,\} = C^1 \times [a,b],$$

so that it coincides with the two special cases. We do not include a or b if they are not finite. We also note that

Remark 4.2. *The proof of Lemma 3.10 depends only on the variable change and the properties of $\cos(X)$, so it is also valid for (4.1) and (4.2).*

Let us now do the phase portrait analysis of (4.2) as we did previously for the other systems.

0- and ∞-isoclines

The $0-$isocline of (4.2) consists of the lines $X = 0$, $X = \pm\pi/2$ and $Y = a$, if included, because this is the only root of G in $[a,b]$ due to its properties. The $\infty-$isocline is implicitly given by

$$k\,M\,G'(Y)\,\cos(X) = \Omega.$$

As in the finite-depth case, this curve is only defined for $X \in (-\pi/2, \pi/2)$ -$G'(Y)$ is always positive-. Moreover, $G''(Y)$ being also positive implies that $G'(Y)$ has a well-defined inverse on $[a,b]$ and thus the $\infty-$isocline can be written as $(X, \gamma(X))$ for $X \in (-\pi/2, \pi/2)$, where

$$\gamma(X) = G'^{-1}\left(\frac{\Omega}{k\,M\,\cos(X)} \right). \tag{4.3}$$

In particular, the curve diverges to $Y \to \infty$ when $X \to \pm\pi/2$ and is of class $\mathcal{C}^2$.

Singular points

The singular points will be given by the intersections of the isoclines that are in $\mathbb{D}$. Let us start with $P_0 = (0, \gamma(0))$. We call $\gamma_0 := \gamma(0)$. If P_0 is to be in the interior of $\mathbb{D}$, then $\gamma_0 \in (a, b)$, and therefore $G(\gamma_0), G''(\gamma_0) > 0$. We now calculate the Hessian matrix of H at P_0:

$$\mathcal{H}_H(P_0) = \begin{pmatrix} H_{XX} & H_{XY} \\ H_{YX} & H_{YY} \end{pmatrix}\Bigg|_{P_0} = kM \begin{pmatrix} -G(Y)\cos(X) & -G'(Y)\sin(X) \\ -G'(Y)\sin(X) & G''(Y)\cos(X) \end{pmatrix}\Bigg|_{P_0} = kM \begin{pmatrix} -G(\gamma_0) & 0 \\ 0 & G''(\gamma_0) \end{pmatrix},$$

so P_0 is non-degenerate and by theorem 3.9 is a saddle point. Thus there will be four separatrices. Below the lowest two, trajectories will go from $X = \pi$ to $X = -\pi$.

Since ultimately we want system (4.2) to be applicable to the physical reality, we must exclude those trajectories that diverge and keep only those ones going from $X = -\pi$ to $X = \pi$, thus, P_0 must lie above $\mathbb{D}$. Therefore, we must impose that the lowest separatrix lies also above $\mathbb{D}$. In particular,

$$G'(b) < \frac{\Omega}{k\,M} \xrightarrow{G''(Y)>0} G'(a) < \frac{\Omega}{k\,M}. \tag{4.4}$$

This second part also implies that there is no other singular point, that is, rejects possible intersections between $Y = a$ and $(X, \gamma(X))$.

4.1.2 Main result

As we did in the previous chapter, for every $Y_\pi \in [a, b]$ we define the period function $\mathcal{T} = \mathcal{T}(Y_\pi)$ as the time needed for the solution of (4.2) with initial condition (π, Y_π) to reach $X = -\pi$. We will now prove the main results of this section.

Theorem 4.3. *There are no periodic orbits in* (4.1)*.*

Proof. Since lemma 3.10 applies, it suffices to prove that $\forall Y_\pi \in [a, b]$, $\mathcal{T}(Y_\pi) > 2\pi/\Omega$. Let $(X(t), Y(t))$ be the solution of (4.2) that has as initial condition (π, Y_π). If the solution intersects the line $X = \pi/2$ at $Y = Y_{\pi/2}$ (and by symmetry also $X = -\pi/2$), from $dY/dt = B(X, Y) = k\,M\,G(Y)\sin(X)$ it is clear that $Y(t)$ lies above $Y = Y_{\pi/2}$ when $X(t) \in (-\pi/2, \pi/2)$ and below when $|X(t)| > \pi/2$. This implies that

$$\frac{dX}{dt} = kMG'(Y)\cos(X) - \Omega \geq kMG'(Y_{\pi/2})\cos(X) - \Omega$$

and thus

$$\mathcal{T}(Y_\pi) \geq \int_{\pi}^{-\pi} \frac{dX}{kMG'(Y_{\pi/2})\cos(X) - \Omega} = \int_{-\pi}^{\pi} \frac{dX}{\Omega - kMG'(Y_{\pi/2})\cos(X)} =$$
$$\frac{1}{\Omega}\int_{-\pi}^{\pi} \frac{dX}{1 - \mu\cos(X)} = \frac{2\pi}{\Omega}\frac{1}{\sqrt{1-\mu^2}} > \frac{2\pi}{\Omega}$$

where the integral of (3.15) was used, provided that

$$\mu := \frac{kM}{\Omega}G'(Y_{\pi/2}) < \frac{kM}{\Omega}G'(\gamma_0) = \frac{kM}{\Omega}\frac{\Omega}{kM} = 1.$$

□

We may, however, go a little bit further in our study of the period function $\mathcal{T}$ and say that it is not only bounded from below, but an increasing function -in [3] it is shown for the deep-water case, we extend it for the general model-. In order to prove this, we must first recall a known result of measure theory (see, for example, [11]):

Theorem 4.4 (***Lebesgue's bounded convergence***)**.** *Let $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions on a measure space (S,Σ,μ). Suppose that the sequence converges point-wise to a function J and is dominated by some integrable function I in the sense that*

$$|J_n(x)| \le I(x), \qquad \forall n\in\mathbb{N},\ \ \forall x\in S. \tag{4.5}$$

Then J is integrable and

$$\lim_{n\to\infty}\int_S J_n\,d\mu = \int_S J\,d\mu.$$

As a consequence

Theorem 4.5. *The period $\mathcal{T}=\mathcal{T}(Y_\pi)$ is an increasing function in (a,b).*

Proof. Given $Y_\pi\in(a,b)$ we have seen that

$$\mathcal{T}(Y_\pi)=\int_{-\pi}^{\pi}\frac{dX}{\Omega-kMG'(Y)\cos(X)}=2\int_0^{\pi}\frac{dX}{\Omega-kMG'(Y)\cos(X)}.$$

It is clear that $\mathcal{T}$ is a differentiable function. Thus, our goal is simply to show that $\mathcal{T}'(Y_\pi)>0$. Let $Y_\pi^n\searrow Y_\pi$ an arbitrary decreasing sequence of values of (a,b). Denote by $Y^n=Y^n(t)$ the second coordinate of the solution of (4.2) with initial condition (π,Y_π^n), which due to the Implicit Function Theorem can be thought as $Y^n=Y^n(X)$. Then,

$$\mathcal{T}'(Y_\pi)=\lim_{n\to\infty}\frac{\mathcal{T}(Y_\pi)-\mathcal{T}(Y_\pi^n)}{Y_\pi-Y_\pi^n}=$$

$$\lim_{n\to\infty}\frac{2}{Y_\pi-Y_\pi^n}\left(\int_0^{\pi}\frac{dX}{\Omega-kMG'(Y)\cos(X)}-\int_0^{\pi}\frac{dX}{\Omega-kMG'(Y^n)\cos(X)}\right)=$$

$$\lim_{n\to\infty}\frac{2}{Y_\pi-Y_\pi^n}\int_0^{\pi}\frac{\Omega-kMG'(Y^n)\cos(X)-\Omega+kMG'(Y)\cos(X)}{[\Omega-kMG'(Y)\cos(X)][\Omega-kMG'(Y^n)\cos(X)]}\,dX=$$

$$2\lim_{n\to\infty}\int_0^{\pi}\frac{kM\cos(X)}{[\Omega-kMG'(Y)\cos(X)][\Omega-kMG'(Y^n)\cos(X)]}\frac{G'(Y)-G'(Y^n)}{Y_\pi-Y_\pi^n}\,dX.$$

Since we are only interested in the sign of this limit, we can multiply its argument by a positive amount, like $(Y_\pi-Y_\pi^n)/(Y-Y^n)$, obtaining

$$2\lim_{n\to\infty}\int_0^{\pi}\frac{kM\cos(X)}{[\Omega-kMG'(Y)\cos(X)][\Omega-kMG'(Y^n)\cos(X)]}\frac{G'(Y)-G'(Y^n)}{Y-Y^n}\,dX$$
$$=:2\lim_{n\to\infty}\int J_n(X)\,dX.$$

We want to apply now theorem 4.4 to the functions $J_n(X)$, but we must first find some dominating integrable function $I(X)$. Let X be fixed. Knowing that $G''(Y)>0$ and is continuous $\forall Y\in[a,b]$, as a consequence of the mean value theorem

$$0<\frac{G'(Y)-G'(Y^n)}{Y-Y^n}=G''(Y^{n,'})\le\mathcal{G}:=\max_{Y\in[a,b]}G''(Y),$$

where $Y^{n,'}$ is some value in $[Y(X),Y^n(X)]\subseteq[a,b]$. We define

$$I(X):=\begin{cases}\dfrac{k\,M}{\Omega-kMG'(Y)\cos(X)}\dfrac{1}{\Omega-kMG'(b)}\dfrac{1}{\mathcal{G}} & \text{if } X\in\left(0,\dfrac{\pi}{2}\right)\\[2ex] \dfrac{k\,M}{\Omega-kMG'(Y)\cos(X)}\dfrac{1}{\Omega}\dfrac{1}{\mathcal{G}} & \text{if } X\in\left(\dfrac{\pi}{2},\pi\right),\end{cases}$$

which is well-defined due to (4.4) and integrable -since it is continuous-. We check now that $|J_n(X)| < I(X)$. On the one hand, for $X \in (0, \pi/2)$, $\cos(X) > 0$ and $\Omega - kMG'(Y^n)\cos(X) > \Omega - kMG'(b) > 0$ so

$$|J_n(X)| = \frac{k\,M}{\Omega - kMG'(Y)\cos(X)}\frac{\cos(X)}{\Omega - kMG'(Y^n)\cos(X)}\frac{G'(Y) - G'(Y^n)}{Y - Y^n} <$$
$$< \frac{k\,M}{\Omega - kMG'(Y)\cos(X)}\frac{1}{\Omega - kMG'(b)}\frac{1}{\mathcal{G}} = I(X).$$

On the other hand, for $X \in (\pi/2, \pi)$, $\cos(X) < 0$ and $\Omega - kMG'(Y^n)\cos(X) > \Omega > 0$ so

$$|J_n(X)| = \frac{k\,M}{\Omega - kMG'(Y)\cos(X)}\frac{|\cos(X)|}{\Omega - kMG'(Y^n)\cos(X)}\frac{G'(Y) - G'(Y^n)}{Y - Y^n} <$$
$$< \frac{k\,M}{\Omega - kMG'(Y)\cos(X)}\frac{1}{\Omega}\frac{1}{\mathcal{G}} = I(X).$$

Finally, continuous dependence on initial conditions implies $Y_n(X) \searrow Y(x)$ so (4.1.2) is equal to

$$2\int_0^\pi \frac{kM\cos(X)}{[\Omega - kMG'(Y)\cos(X)]^2}\lim_{n\to\infty}\frac{G'(Y) - G'(Y^n)}{Y - Y^n}\,dX = 2\int_0^\pi \frac{kM\cos(X)G''(Y)}{[\Omega - kMG'(Y)\cos(X)]^2} =$$
$$2\int_0^{\pi/2} \frac{kM\cos(X)G''(Y)}{[\Omega - kMG'(Y)\cos(X)]^2} + 2\int_{\pi/2}^{\pi} \frac{kM\cos(X)G''(Y)}{[\Omega - kMG'(Y)\cos(X)]^2} \geq$$
$$2\int_0^{\pi/2} \frac{kM\cos(X)G''(Y_{\pi/2})}{[\Omega - kMG'(Y)\cos(X)]^2} + 2\int_{\pi/2}^{\pi} \frac{kM\cos(X)G''(Y_{\pi/2})}{[\Omega - kMG'(Y)\cos(X)]^2} >$$
$$2kMG''(Y_{\pi/2})\left(\int_0^{\pi/2}\frac{\cos(X)}{\Omega^2}\,dX + \int_{\pi/2}^{\pi}\frac{\cos(X)}{\Omega^2}\,dX\right) = 0.$$

□

In Fig. 4.1 we can see the period function $\mathcal{T}$ for the finite-depth function $G(Y) = \sinh(Y)$.

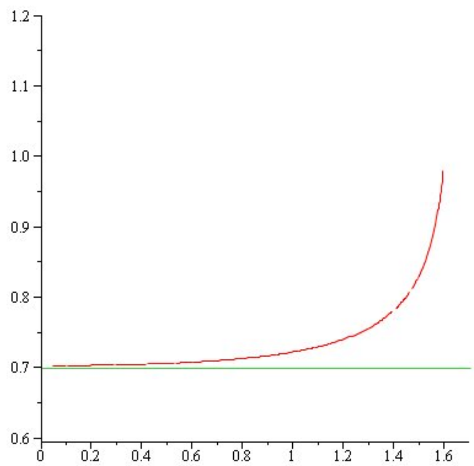

Figure 4.1: *Numerical approximation of the period function* $\mathcal{T} = \mathcal{T}(Y_\pi)$ *(red) using Maple®. The green line represents the value* $2\pi/\Omega$. *In this example* $\Omega = 9$.

4.2 Some modified systems

So far we have seen that the equations of motion of the particle paths in irrotational linear water waves, as well as for its more general model, have no periodic solution and its period function is increasing. However, we may now wonder -for purely mathematical purposes- whether it is possible to modify (4.1) in some way so that it does have periodic orbits, or a qualitatively different period function. We see that (4.1) is of the form

$$\begin{cases} \dfrac{dx}{dt} = M\,G'(ky)\,F(kx-\Omega t) \\ \dfrac{dy}{dt} = M\,G(ky)\,F'(kx-\Omega t), \end{cases} \tag{4.6}$$

where $F(x)=\cos(x)$ is a $2\pi-$periodic function. A natural way to proceed is to consider either functions $G(Y)$ with slightly different conditions than the ones stated at the beginning of §4.1.1, or periodic functions $F(X)$ other than $\cos(X)$.

In this section three examples of such modifications will be given: the first, somewhat trivial, with a constant period function although with no periodic orbits, the second, more elaborated, for which existence of periodic orbits has been proven, and the third, a combination of the previous two, which leads to a whole set of periodic orbits.

4.2.1 A case of constant period function

The first example takes exactly the same form of (4.1), (4.2) but allowing $G''(Y)=0$, so that $G(Y)=Y$ can be chosen. If we want the first condition on G to be fulfilled, we must set $a=0$. Our conjugated systems will be

$$\begin{cases} \dfrac{dx}{dt} = M\,\cos(kx-\Omega t) \\ \dfrac{dy}{dt} = kMy\sin(kx-\Omega t), \end{cases} \tag{4.7}$$

and

$$\begin{cases} \dfrac{dX}{dt} & = k\,M\,\cos(X)-\Omega & := A(X,Y) \\ \dfrac{dY}{dt} & = k\,M\,Y\sin(X) & := B(X,Y) \end{cases} \tag{4.8}$$

which is also Hamiltonian:

$$H(X,Y) = k\,M\,Y\,\cos(X) - \Omega\,Y = (k\,M\,\cos(X)-\Omega)Y.$$

The phase portrait analysis of this system is easy to do. In first place, we see that the 0-isocline is formed, as in the previous section, by the lines $X=0$, $X=\pm\pi/2$ and $Y=0$. The main difference is in the $\infty-$isocline, that now does not depend on Y, and are the lines $X=\arccos(\Omega/kM)$, which of course only exist if $\Omega\leq kM$. However, if the $\infty-$isocline contains vertical lines, these are going to be invariant manifolds ([6]) and in that case no trajectory would go from $X=\pi$ to $X=-\pi$. All would diverge. Therefore, we must impose $\mu:=\Omega/kM<1$. In that case, no singular points are to be found. Whatismore, since $B(X,Y)=B(X)$, the period function can be explicitly calculated

$$\mathcal{T}(Y_\pi) = \int_{-\pi}^{\pi}\frac{dX}{\Omega-kM\cos(X)} = \frac{1}{\Omega}\int_{-\pi}^{\pi}\frac{dX}{1-\mu\cos(X)} = \frac{1}{\Omega}\frac{2\pi}{\sqrt{1-\mu^2}},$$

which is independent of Y_π and greater than $2\pi/\Omega$, so by lemma 3.10 there are no periodic orbits of (4.7). In Fig. 4.2 we can see the phase portrait of (4.8).

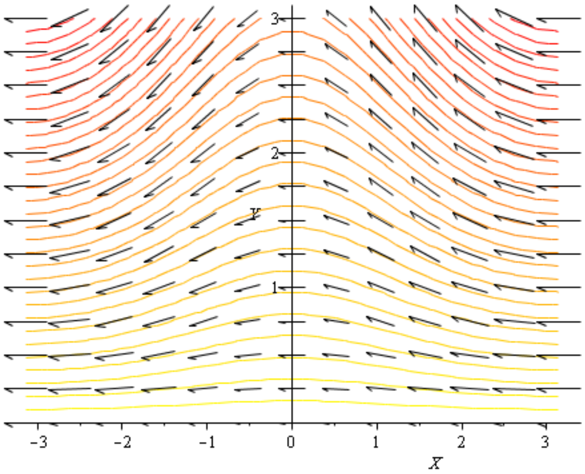

Figure 4.2: *Phase portrait of system* (4.8)*. All trajectories go from* $X = \pi$ *to* $X = -\pi$*.*

4.2.2 A perturbed system with periodic orbits

Another interesting case is when we perturb the system (4.2) with a parameter $\alpha \in (0,1)$ so that $\cos(X) \mapsto \cos(X) - \alpha$. In this case we will have

$$\begin{cases} \dfrac{dx}{dt} = M\, G'(ky)\,[\cos(kx - \Omega t) - \alpha] \\ \dfrac{dy}{dt} = M\, G(ky)\, \sin(kx - \Omega t), \end{cases} \tag{4.9}$$

and

$$\begin{cases} \dfrac{dX}{dt} & = k\, M\, G'(Y)\,[\cos(X) - \alpha] - \Omega & := A(X,Y) \\ \dfrac{dY}{dt} & = k\, M\, G(Y)\, \sin(X) & := B(X,Y) \end{cases} \tag{4.10}$$

which is also Hamiltonian:

$$H(X,Y) = k\, M\, G(Y)\,[\cos(X) - \alpha] - \Omega\, Y.$$

In this case we will impose a finite. The $0-$isocline is the usual: the lines $X = 0$, $X = \pm\pi/2$ and $Y = a$, and the $\infty-$isocline will be given by a curve of the form $(X, \gamma(X))$, similar to (4.3) but this time defined only for $X \in (-X_\alpha, X_\alpha)$, being $X_\alpha = \arccos(\alpha) \le \pi/2$, and given by

$$\gamma(X) = {G'}^{-1}\left(\frac{\Omega}{k\, M\,[\cos(X) - \alpha]}\right),$$

which diverges to $Y \to \infty$ when $X \to \pm X_\alpha$.

Regarding the singular points, we have a similar situation to the one we had for system (4.2). We first consider the point $P_0 = (0, \gamma(0)) = (0, \gamma_0)$, where $\gamma_0 := \gamma(0)$. If P_0 lies in the interior of $\mathbb{D}$, then $\gamma_0 \in (a,b)$, so $G(\gamma_0), G''(\gamma_0) > 0$ and the Hessian matrix of H in P_0 will be

$$\mathcal{H}_H(P_0) = \begin{pmatrix} H_{XX} & H_{XY} \\ H_{YX} & H_{YY} \end{pmatrix}\Bigg|_{P_0} = kM \begin{pmatrix} -G(\gamma_0) & 0 \\ 0 & G''(\gamma_0)[1-\alpha] \end{pmatrix},$$

so P_0 is again non-degenerate and by theorem 3.9 is a saddle point. There will be four separatrices and below the lowest two, trajectories will go from $X = \pi$ to $X = -\pi$. Just as we did before, we leave this point above our domain, so that

$$G'(a) < G'(b) < \frac{\Omega}{kM(1-\alpha)}. \tag{4.11}$$

With this all other possible intersections between $Y = a$ and $(X, \gamma(X))$ are rejected, so there are no other singular points. All trajectories go now from $X = \pi$ to $X = -\pi$.

Existence of periodic orbits

We can now give a condition for the existence of periodic orbits.

Theorem 4.6. *If*

$$\frac{\alpha}{1-\alpha^2} > \frac{kMG'(a)}{2\Omega}, \tag{4.12}$$

and b is large enough, then (4.10) *has at least one periodic orbit.*

Proof. A first observation to be made is that we can always find values of α fulfilling (4.12), because

$$\lim_{\alpha\to 0+} \frac{\alpha}{1-\alpha^2} = 0 \qquad \lim_{\alpha\to 1-} \frac{\alpha}{1-\alpha^2} = +\infty$$

and the function is continuous therein. In order to understand better the proof, we look at Fig. 4.3.

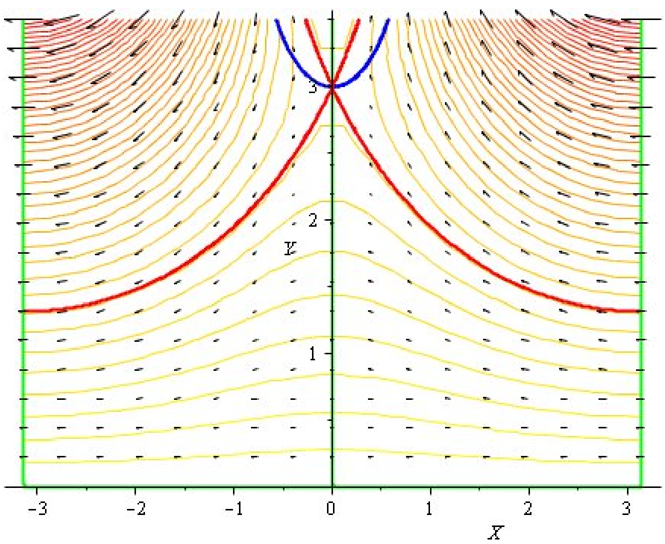

Figure 4.3: *Phase portrait of system* (4.9). *As usual, the 0-isocline and $\infty-$isocline are green and blue respectively, and the separatrices are represented in red colour.*

We see that as Y_π increases, the trajectories get closer to the separatrix. Since the separatrix, understood as a trajectory, takes an infinite amount of time to get from $X = \pi$ to $X = 0$, and the period function $\mathcal{T} = \mathcal{T}(Y_\pi)$ is a continuous function of Y_π, for large enough b the period will be arbitrarily long.

However, if $Y_\pi = a$, then $dY/dt = 0$ and the trajectory is flat. Therefore, the period function in that point can be computed explicitly

$$\mathcal{T}(a) = \int_{-\pi}^{\pi} \frac{dX}{\Omega - kMG'(a)[\cos(X) - \alpha]} = \frac{1}{\Omega + \alpha kMG'(a)} \int_{-\pi}^{\pi} \frac{dX}{1 - \dfrac{kMG'(a)}{\Omega + \alpha kMG'(a)} \cos(X)} =$$

$$\frac{1}{\Omega + \alpha kMG'(a)} \frac{2\pi}{\sqrt{1 - \left(\dfrac{kMG'(a)}{\Omega + \alpha kMG'(a)}\right)^2}} = \frac{2\pi}{\sqrt{[\Omega + \alpha kMG'(a)]^2 - [kMG'(a)]^2}} =$$

$$\frac{2\pi}{\sqrt{\Omega^2 + 2\Omega\alpha kMG'(a) + (\alpha^2 - 1)k^2M^2G'^2(a)}} < \frac{2\pi}{\Omega},$$

where we used condition (4.12) in the last step. $\mathcal{T}$ is less that the amount required by lemma 3.10, so by the continuity of $\mathcal{T}$ there must be a Y_π such that $\mathcal{T}(Y_\pi) = 2\pi/\Omega$ and therefore correspond through the variable change (3.8) to a periodic orbit of the system (4.9). □

4.2.3 Mixing the two systems

In the first particular case we have seen a system for which the period function was constant, although no periodic orbits were to be found. In the second, existence of periodic orbits was shown, although not much was known about the period function. These findings suggest a new system, for which the period function will also be constant, and with the value required by lemma 3.10, so that all the orbits are periodic.

Let us consider the systems

$$\begin{cases} \dfrac{dx}{dt} = M\,[\cos(kx - \Omega t) - \alpha] \\ \dfrac{dy}{dt} = M\,k\,y\,\sin(kx - \Omega t), \end{cases} \tag{4.13}$$

and

$$\begin{cases} \dfrac{dX}{dt} & = k\,M\,[\cos(X) - \alpha] - \Omega & := A(X,Y) \\ \dfrac{dY}{dt} & = k\,M\,Y\,\sin(X) & := B(X,Y) \end{cases} \tag{4.14}$$

which is also Hamiltonian:

$$H(X,Y) = k\,M\,Y\,[\cos(X) - \alpha] - \Omega\,Y = \{k\,M\,[\cos(X) - \alpha] - \Omega\}\,Y.$$

These systems are very similar to the ones we dealt with in §4.2.1. We must set $a = 0$. The 0-isocline is again the lines $X = 0$, $X = \pm\pi$ and $Y = a = 0$, while the ∞−isocline, again independent of Y, are the vertical lines

$$X = \pm\arccos\left(\frac{\Omega + \alpha kM}{kM}\right),$$

again an invariant manifold. Since we do not want these lines to exist, we must impose

$$\frac{1}{\mu} := \frac{\Omega + \alpha kM}{kM} > 1. \tag{4.15}$$

If the ∞−isocline does not exist, then there will be no singular points: all trajectories of (4.14) will go from $X = \pi$ to $X = -\pi$ in finite time (qualitatively is the same as in §4.2.1 so we can also look at Fig. 4.2). Actually we have,

Theorem 4.7. *If*

$$\frac{\alpha}{1-\alpha^2} = \frac{kM}{2\Omega} \qquad \text{or equivalently} \qquad \alpha = \frac{\sqrt{1+4\beta^2}}{2\beta}, \quad \beta := \frac{kM}{2\Omega}, \tag{4.16}$$

then all trajectories of system (4.13) *are periodic.*

Proof. The period function $\mathcal{T}(Y_\pi)$ of system (4.14) can be calculated explicitly from its first equation:

$$\mathcal{T}(Y_\pi) = \int_{-\pi}^{\pi} \frac{dX}{\Omega - kM[\cos(X) - \alpha]} = \int_{-\pi}^{\pi} \frac{dX}{\Omega - kM[\cos(X) - \alpha]} =$$
$$\frac{1}{\Omega - \alpha kM} \int_{-\pi}^{\pi} \frac{dX}{1 - \mu\cos(X)} = \frac{1}{\Omega - \alpha kM} \frac{2\pi}{\sqrt{1 - \left(\dfrac{kM}{\Omega - \alpha kM}\right)^2}} =$$
$$\frac{2\pi}{\sqrt{\Omega^2 + 2\Omega\alpha kM - (1-\alpha^2)k^2M^2}} = \frac{2\pi}{\Omega},$$

where condition (4.15) was used in the last step. This amount is independent of Y_π, and by lemma 3.10 correspond to periodic trajectories or closed paths of system (4.13) through the variable change (3.8). □

Other possibilities

In this section we only considered systems of the form (4.6). However, the variable change (3.8) can be also used with other kinds of systems, which may have periodic orbits too. For example, in [7] M. Ehrnström and G. Villari study linear gravitational water waves with constant vorticity and get the system

$$\begin{cases} \dfrac{dx}{dt} = M\cosh(ky)\cos(kx - \Omega t) - \omega y \\ \dfrac{dy}{dt} = M\sinh(ky)\sin(kx - \Omega t), \end{cases} \tag{4.17}$$

where ω is the vorticity. They show that for positive and large enough ω the system has periodic orbits indeed. The study of possible modifications of the vorticity model could lead to another work similar to the present.

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Resum del Treball en Català

Les ones són un dels fenòmens més estudiats tant dins de les matemàtiques com de la física. És ben sabut que si llancem una pedra dins d'un llac, o mirem les onades del mar des de la platja, allò que veiem moure's no és pas l'aigua, contràriament a la nostra primera impressió, sinó un patró, una pertorbació que es trasllada a través d'ella. Dit d'una altra manera, el moviment ràpid de les ones és el producte d'un moviment molt més lent de la substància a través de la qual es transmet.

En aquest treball considerem únicament *ones gravitacionals d'aigua*, és a dir, les ones que es formen a l'aigua sota una força gravitatòria constant. Com a punt de partida presentem les hipòtesis generals de la física de fluids i deduïm les equacions del moviment d'Euler per a fluids sense viscositat, així com l'equació de conservació de la massa. Seguidament afegim unes condicions de contorn convenients per a l'estudi d'ones d'aigua i fem una adimensionalització per a una major comoditat en el tractament matemàtic. Després apliquem un canvi d'escala per a restringir-nos al règim d'ones lineals, és a dir, aquelles per a les quals el *paràmetre d'amplitud* -és a dir, el quocient de l'amplada de les ones per la profunditat de l'aigua- és petit.

Ens interessen únicament *solucions viatgeres*, és a dir, solucions en què la dependència espai-temporal és de la forma $x - ct$, on c és la velocitat de propagació de l'ona. Imposant aquesta condició trobem una forma explícita per al camp de velocitats del fluid $\mathbf{u}(\mathbf{x}, \mathbf{t})$. Tot i que des del punt de vista de la hidrodinàmica podríem considerar que amb això ja hem acabat la descripció de l'estat del fluid, hi ha certes qüestions que no queden plenament resoltes. Una d'elles és si les trajectòries de les partícules del fluid, és a dir, les solucions de l'equació $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, \mathbf{t})$ són tancades o no.

Aquesta equació és, de fet, un sistema completament no lineal i no autònom d'equacions diferencials, per al qual no disposem de solucions explícites -almenys sota la hipòtesi d'irrotacionalitat-. L'enfocament clàssic a aquest problema -que podem trobar a la majoria de llibres de mecànica de fluids- consisteix a linealitzar les equacions, obtenint com a solució trajectòries el·líptiques en el cas de profunditat finita, o circulars en el cas d'aigües profundes, és a dir, trajectòries tancades. No obstant això, G. G. Stokes ja va observar l'any 1847 que:

> *"Sembla que el moviment cap endavant de les partícules no és completament compensat pel moviment cap enrere; de manera que, a part del seu moviment oscil·latori, les partícules presenten un moviment progressiu en la direcció de propagació de les ones."*

En els darrers anys s'han publicat diversos avenços en aquest tema de la mà d'A. Constantin *et. al.* i que donen la raó a Stokes, és a dir, que hi ha efectivament un desplaçament net cap endavant -seguint la direcció de propagació de les ones-, que anomenem *deriva de Stokes* (de l'anglès *Stokes drift*). Aquests resultats es basen en un ocurrent canvi de variables que transforma el sistema d'equacions en autònom -i de fet Hamiltonià-, de manera que les eines de l'anàlisi de retrats de fase i de corbes implícites es posen al nostre abast. El nou sistema té un espai de fase cilíndric, per al qual totes les òrbites són periòdiques -al voltant del cilindre-. Així doncs, podem definir-hi una *funció de període*, que per a cada condició inicial ens digui el temps que triga en donar una volta. Es demostra que les òrbites del primer sistema seran tancades si i només si la funció període coincideix amb una certa cota. Finalment, es prova que aquesta cota no s'assoleix mai.

En aquest treball presentem i comentem aquests resultats per al cas d'aigua irrotacional, tant per a profunditat finita com per a aigües profundes. A continuació incloem un capítol amb el treball original, que dividim en dues parts. A la primera veiem que els resultats similars que s'obtenen per a les dues profunditats es poden entendre com a casos particulars d'un model més general, per al qual extenem els resultats de no existència d'òrbites periòdiques, així com de monotonia de la funció període. A la segona intentem trobar sistemes no autònoms similars per als quals, usant la mateixa tècnica, poguem provar que tenen òrbites periòdiques. Donem tres exemples: un amb funció de període constant però per sobre de la cota, un altre que assoleix la cota però té una funció de període no constant, i finalment un amb funció de període constant i que coincideix amb la cota, de manera que totes les seves òrbites són periòdiques.

Resumen del Trabajo en Español

Las ondas son uno de los fenómenos más estudiados tanto dentro de las matemáticas como de la física. Es bien sabido que si lanzamos una piedra en un lago, o miramos las olas del mar desde la playa, lo que vemos moverse no es el agua, contrariamente a nuestra primera impresión, sino un patrón, una perturbación que se traslada a través del agua. Dicho de otra forma, el rápido movimiento de las ondas es el producto de un movimiento mucho más lento de la substancia a través de la cual se transmite.

En este trabajo consideramos únicamente *ondas gravitacionales de agua*, es decir, ondas que se forman en el agua bajo una fuerza gravitatoria constante. Como punto de partida presentamos las hipótesis generales de la física de fluidos y deducimos las ecuaciones del movimiento de Euler para fluidos sin viscosidad, así como la ecuación de conservación de la masa. Seguidamente añadimos unas convenientes condiciones de contorno para el estudio de ondas de agua y realizamos una adimensionalización para mayor comodidad en el tratamiento matemático. Después aplicamos un cambio de escala para restringirnos al régimen de ondas lineales, es decir, aquellas cuyo *parámetro de amplitud* -el cociente de la amplitud de las ondas por la profundidad del agua- es pequeño.

Nos interesan únicamente *soluciones viajeras*, es decir, soluciones cuya dependencia espacio-temporal es de la forma $x - ct$, donde c es la velocidad de propagación de la onda. Imponiendo esta condición hallamos una forma explícita del campo de velocidades del fluido $\mathbf{u}(\mathbf{x}, \mathbf{t})$. Aunque desde el punto de vista de la hidrodinámica podríamos considerar que con esto ya hemos acabado la descripción del fluido, hay ciertas cuestiones que no quedan plenamente resueltas. Una de ellas es si las trayectorias de las partículas del fluido, es decir, las soluciones de la ecuación $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, \mathbf{t})$, son cerradas o no.

Esta ecuación es, de hecho, un sistema completamente no lineal y no autónomo de ecuaciones diferenciales, para el cual no disponemos de soluciones explícitas -al menos bajo hipótesis de irrotacionalidad-. El enfoque clásico para este problema -que podemos encontrar en la mayoría de libros de mecánica de fluidos- consiste en linealizar las ecuaciones, obteniendo como solución trayectorias elípticas en el caso de profundidad finita, o circulares en el caso de aguas profundas, es decir, trayectorias cerradas. No obstante, G. G. Stokes ya observó en 1847 que:

> *"Al parecer el movimiento hacia adelante de las partículas no es totalmente compensado por el movimiento hacia atrás; de manera que, a parte de su movimiento oscilatorio, las partículas presentan un movimiento progresivo en la dirección de propagación de las ondas."*

En los últimos años se han publicado varios avances en este tema de la mano de A. Constantin *et. al.* y que dan la razón a *Stokes*, es decir, que hay efectivamente un desplazamiento neto hacia adelante -siguiendo la dirección de propagación de las ondas-, que llamamos *deriva de Stokes* (del inglés *Stokes drift*). Estos resultados se basan en un ocurrente cambio de variables que transforma el sistema de ecuaciones en autónomo -y de hecho Hamiltoniano-, de manera que las herramientas del análisis de retratos de fase y de curvas implícitas se ponen a nuestro abasto. El nuevo sistema tiene un espacio de fase cilíndrico, para el cual todas las órbitas son periódicas -alrededor del cilindro-. Esto permite definir una *función de periodo* que para cada condición inicial nos diga el tiempo que tarda en dar una vuelta. Se demuestra que las órbitas del primer sistema seran cerradas si y solo si la función periodo coincide con cierto valor crítico. Finalmente se prueba que nunca se llega a este valor crítico.

En este trabajo presentamos y comentamos estos resultados para el caso de agua irrotacional, tanto en produndidad finita como en aguas profundas. Seguidamente incluimos un capítulo que contiene el trabajo original y que dividimos en dos partes. En la primera vemos que los resultados similares que se obtienen para las dos profundidades se pueden entender como casos particulares de un modelo más general, para el cual extendemos los resultados de no existencia de órbitas periódicas, así como la monotonía de la función periodo. En la segunda intentamos encontrar sistemas no autónomos similares para los cuales, usando la misma técnica, podamos probar que tienen órbitas periódicas. Damos tres ejemplos: uno con una función de periodo constante pero por sobre del valor crítico, otra que llega al valor crítico pero con una función de periodo no constante, y finalmente una con función de periodo constante y que coincide con el valor crítico, de manera que todas sus órbitas son periódicas.

Deutsche Zusammenfassung

Die Wellen gehören zu den Phänomenen, die sowohl in der Mathematik als auch in der Physik am meisten studiert werden. Wenn wir einen Stein ins Meer werfen so wissen wir, dass die Bewegung, welche wir sehen, nicht das Wasser selbst (im Gegensatz zu unserem ersten Eindruck), sondern eine sich durch das Wasser bewegende Störung ist. Mit anderen Worten, die schnelle Wellenbewegung ist nichts anders als das Produkt der viel langsameren Bewegung des Wassers.

In dieser Arbeit beschränken wir uns auf *Wasserwellen*, das heißt, Wellen die sich in Wasser unter Einwirkung vom Schwerkraft bilden. An den Ausgangspunkt stellen wir die Hypothesen der Strömungsphysik und leiten sowohl die Euler-Gleichungen als auch den Massenerhaltungssatz daraus her. Nachher stellen wir passende Randbedingungen und machen das System dimensionslos -damit die mathematischer Handlung einfacher ist-. Danach skalieren wir einige Variablen, um nur mit linearen Wellen zu arbeiten, das heißt, dass der Amplitudeparameter -die Schwingungsweite durch die Wassertiefe- klein ist.

Wir fokussieren uns auf Reiselösungen, das sind, Lösungen, deren Raumzeitabhängigkeit der Form $x - ct$ sind, wo c die Wellengeschwindigkeit ist. Wenn wir das durchsetzen, finden wir das Geschwindigkeitsfeld $\mathbf{u}(\mathbf{x}, \mathbf{t})$ der Flüssigkeit explizit. Die Beschreibung des Flüssigkeitszustands ist aber damit noch nicht vollständig, sondern einige Fragen bleiben unbeantwortet. Zum Beispiel, ob die Trajektorien der Flüssigkeitsteilchen -die Lösungen der Gleichung $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, \mathbf{t})$- geschlossen sind.

Diese Gleichung ist eigentlich ein nichtlineares zeitabhängiges Differenzialgleichungssystem und explizite Lösungen stehen nicht zur Verfügung -wenigstens wenn die Wirbelstärke verschwindet-. Der gewöhnliche Ansatz -welcher in den meisten Büchern gefunden werden kann- besteht darin, das Gleichungssystem zu linearisieren. Damit bekommt man elliptische Trajektorien für eine endliche Tiefe, und kreisförmige Trajektorien für Tiefwasser, die in beiden Fällen geschlossen sind. Nichtsdestoweniger, schon im Jahr 1847 beobachtete G. G. Stokes:

> *,,Anscheinend wird die Vorwärtsbewegung der Teilchen nicht durch ihre Rückwärtsbewegung völlig ausgegliechen; so dass die Teilchen eine fortschreitende Bewegung in der Ausbreitungsrichtung zusätzlich zu der Schwingungsbewegung haben."*

In den letzten Jahren wurden einige Entwicklungen von A. Constantin u. a. veröffentlicht, die Stokes Recht geben, das heißt, dass es tatsächlich eine Nettovorwärtsbewegung -in der Ausbreitungsrichtung- gibt, welche *Stokes Drift* gennant wird. Das wird durch einen einfallsreichen Variablenwechsel bewiesen, der das zeitabhängiges System autonom -und tatsächlich zu einem Hamiltonschen System- macht. Damit stellt man die Möglichkeit zur Verfügung, nicht nur eine Phasenraumanalyse zu machen, sondern auch Kurvengeometrie zu nutzen. Das neue System hat einen zylindrischen Phasenraum, wo alle Trajektorien geschlossen um den Zylinder und periodisch sind. Damit lässt sich eine Periodenfunktion definieren. Man kann beweisen, dass eine Trajektorie aus dem ersten System periodisch genau dann ist, wenn die Periode der entsprechenden Trajektorie aus dem zweiten System ein bestimmtes Wert annimmt. Schließlich zeigt man, dass das niemals der Fall ist.

In dieser Arbeit sollen diese Ergebnisse für Wirbelstärkeloses Wasser vorgestellt und besprochen werden; sowohl für endliche Tiefen als auch für Tiefwasser. Danach kommt die eigentliche Arbeit, die aus zwei Teilen besteht: zuerst wird gezeigt, dass die ähnliche Ergebnisse, welche man in beiden Fällen bekommt, als Sonderfälle eines allgemeinen Modells verstanden werden können, auf den die Ergebnisse ausgebreitet werden. Im zweiten Teil versuchen wir, ähnliche zeitabhängige Differenzialgleichungssysteme zu finden, für die die Existenz periodischer Trajektorien mit selben Technik bewiesen werden kann. Drei Beispiele werden gegeben: eines, mit einer konstanten Periodenfunktion, das aber keine periodische Trajektorie hat, eines, mit einer nichtkonstanten Periodenfunktion aber mit periodischen Trajektorien, und eines, mit nur periodischen Trajektorien.

Acknowledgements

I would like to thank my supervisor Prof. A. Gasull for his time and stimulating conversations, as well as A. Schneider for the german corrections. Furthermore, I wish to thank my family for their constant support throughout my studies.